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Topic 07: Equilibrium and Continuity

In this topic we look at *equilibrium* (sv: *jämviktsläge*) solutions of circuits that include capacitors and inductors as well as the resistors and sources that we already know. Then the concept of *continuity* is used to find solutions of these circuits immediately after the equilibrium is disturbed by a switch or a change of component-value.

These analyses are useful in their own right, since equilibria and sudden changes are frequently of interest. They are also a necessary step in finding *initial conditions* (sv: *begynnelsevärde*) for differential equations in the next topic.

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Let's have a reminder of the fundamental equations of the capacitor and inductor, in derivative form, with a comparison to resistance. The reference directions of i and u have been chosen so that negative signs are not needed in the equations.

$$u(t) = R\, i(t) \tag{1}$$

$$i(t) = C\,\frac{\mathrm{d}u(t)}{\mathrm{d}t} \tag{2}$$

$$u(t) = L\,\frac{\mathrm{d}i(t)}{\mathrm{d}t} \tag{3}$$

Now, consider the integral form of the same equations:

$$i(t) = \frac{1}{R}\,u(t) \tag{4}$$

$$u(t) = u(t_0) + \frac{1}{C}\int_{t_0}^{t} i(x)\,\mathrm{d}x \tag{5}$$

$$i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} u(x)\,\mathrm{d}x. \tag{6}$$

What is special about the capacitor and inductor, compared to components we saw previously, is that as well as their *component value* of (capacitance or inductance), they have at any time a particular *state* of stored energy, related to a value of the capacitor's voltage or the inductor's current.

In this way the reactive components have *two* important properties to consider when doing circuit calculations: the component value, *and* the state. It turns out that for solving a circuit's voltages and currents *at* a particular instant in time the capacitors' and inductors' states are important but their component values are irrelevant. However, the component values are important for calculating how quickly the state is *changing* with time, or in other words, for finding what the state will be at a later time.

1 Equilibrium

Consider a circuit with sources, resistors, capacitors and inductors, where all sources and other components have constant values, and the circuit has been in this situation for "a very long time". We then assume that all the currents and voltages will have reached *equilibrium* values that are constant.[1]

1.1 Does an equilibrium exist?

The above *assumption* is clearly *not* a certainty in the world of idealised circuits. For example, an inductor connected in parallel with a constant voltage source will have a current that just keeps changing forever, $\frac{\mathrm{d}i(t)}{\mathrm{d}t} = U/L =$ constant! Even worse, an inductor or capacitor might be connected to a circuit that behaves as a Thevenin or Norton source with negative resistance: this could result in a change, $\frac{\mathrm{d}i}{\mathrm{d}t}$ or $\frac{\mathrm{d}u}{\mathrm{d}t}$, that keeps getting *bigger* with time, instead of letting the circuit settle to a steady value.

But such cases are really just amusements found in idealised circuits. In practical cases there will be some resistance between and within the components. This will limit how large the capacitor voltages and inductor currents can become. If an inductor is connected to a voltage source through a resistor, the current will not increase beyond the level where the full voltage of the source is being used to keep pushing this current through the series resistance; at this point, KVL tells us there is no voltage across the inductor to cause a further change of current. If a capacitor is connected to a current source with a parallel resistor, the voltage will not increase beyond the level at which all the source current passes through the resistor and therefore is not charging the capacitor.

Idealised circuits, particularly when including controlled sources and negative resistance, are usually only good approximations of real circuits within a moderate range of the circuit quantities. If a voltage or current keeps increasing, then resistances that seemed negligible at low voltages and currents will have to be included in the model, and opamps or transistor outputs will reach supply-voltage limits. In most power-oriented calculations we wouldn't have negative resistance in the model in the first place. The only plausible way for a real circuit to fail to reach an equilibrium is if some circuit quantities *oscillate* instead of reaching a steady value; a permanent oscillation in a linear circuit with constant sources and non-zero resistance requires some sort of controlled source.

We can therefore justifiably make the assumption of steady final values of circuit quantities, if a circuit contains resistances between other components and

[1] As we are interested in practical things rather than mathematical curiosities, a value that is "approaching an asymptotic value" can be regarded as having reached the equilibrium when it's closer than some limit that we can choose.

does not contain dependent sources. Many other circuits even without so many resistors or including controlled sources can also reach steady equilibrium values of voltages and current.

1.2 Principle of equilibrium solution

Can we, with our existing knowledge, calculate all the currents and voltages in an equilibrium state, even when there are capacitors and inductors in the circuit? Will this require complicated new methods beyond dc analysis? (Answers: Yes and No, respectively!)

We are considering a long time with constant values of sources, where all circuit quantities have reached an equilibrium of steady values: this means that $\frac{\mathrm{d}u}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}i}{\mathrm{d}t} = 0$ for *all* the voltages and currents in the circuit. (Remember the warning in the previous section about whether such an equilibrium exists: in idealised circuits it might not.)

From the equations defining a capacitor and inductor (2) (3), we see that if $\frac{\mathrm{d}u}{\mathrm{d}t} = 0$ for a capacitor, then the current in that capacitor must be zero, and similarly that if $\frac{\mathrm{d}i}{\mathrm{d}t} = 0$ for an inductor then that inductor's voltage must be zero.

This shows how to handle equilibrium calculations. For a capacitor, no *current* flows in it in equilibrium: this means we can consider it a current source of zero, or more simply an *open circuit*. For an inductor, there is no *voltage* across it in equilibrium: we can consider it as a voltage-source of zero, or more simply a *short circuit*.

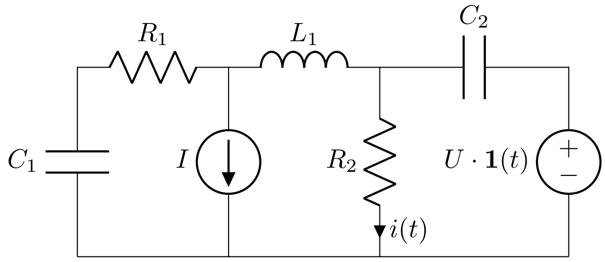
In this way, calculation of equilibrium just requires us to replace capacitors and inductors with open- and short-circuits respectively, and to calculate currents and voltages in the circuit. The solution will let us find the capacitors' voltages and inductors' currents, which were the unknowns for the open and short circuits.

This is very easy! It uses just our existing knowledge of dc circuit solutions. Replacement of components with open or short circuits results in a much simplified circuit-diagram. If you are given a pure equilibrium problem to solve, then seeing lots and lots of capacitors and inductors probably means it will be *easier* than if lots of these were resistors and sources instead: the capacitors and inductors will all turn into simple open- and short-circuits for the analysis. (But in Topic 08 about time-functions you should be very scared if you see lots of capacitors or inductors; we will probably not even handle more than one at a time for that topic!)

1.3 Example of equilibrium solution

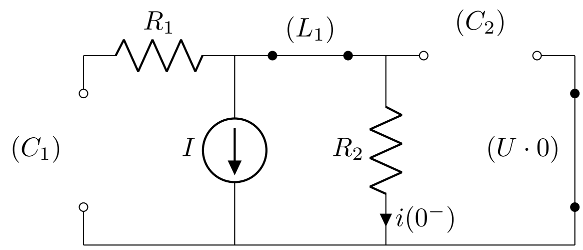
The following circuit provides a simple example of equilibrium calculation in a circuit with several reactive components. Only one change happens, by the voltage source having a step at $t = 0$; at times before and after this, the circuit has constant source-values and no switches.

Only one variable is being sought: $i(t)$ down resistor R_2. If you want to make it more interesting, then define some other unknowns to solve for, such as the voltage across C_1 or current in L_1.



There are *two* equilibria we could look at. The time $t = 0^-$ is just before the voltage step. It is assumed by default that the circuit has been standing with the same source values "ever since $t = -\infty$", as nothing else was said. We can therefore assume that there is an equilibrium condition at time $t = 0^-$.[2] Just after $t = 0$, the earlier equilibrium has been disturbed by the voltage step, so the equilibrium assumption is not true. But after a very long time, $t \to \infty$, another equilibrium is reached.

In order to solve for the marked current at the initial equilibrium, $i(0^-)$, we can redraw this circuit for time $t = 0^-$ in the simplest possible way. All reactive components can be replaced by open or short circuits, on the assumption of a constant equilibrium state. The voltage source is set to zero (short-circuit) because $\mathbf{1}(0^-) = 0$.



After this redrawing it is very clear what the sought current must be! There is only one possible path for the current from the current source: $i(0^-) = -I$.

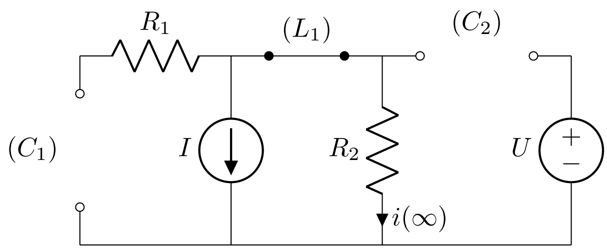
Any other variables could also have been found. For example, one variable of each reactive component was already known to be zero due to the equilibrium, but the other variables can now be found too. The current in L_1 is also I (plus or minus, depending on which way you choose to define it). The voltages across C_2 and across C_1 are the same as across R_2, which is IR_2.[3]

[2] We don't assume equilibrium to have been finally reached *at* $t = 0^-$; if the circuit has stood there 'always' then it's assumed to have reached equilibrium 'ages ago'. Note again that there's nothing special about $t = 0$; this is often used as the definition of the time when a single change happens, but one could instead call it $t = t_x$ or t_1 and write $U \cdot 1(t - t_x)$ for the source.

[3] That warning again: be careful about the current source: remember that its voltage is *not known* until we calculate it based on what voltage is needed in order to force its current through the rest of the circuit ... it should not be assumed to be zero.

A similar process is, of course, used to study the other equilibrium for this circuit, when $t \to \infty$. The difference is only the voltage source, which is now active, with a value of U.

Like the wise circuit-analysts that we are, we re-draw the original diagram carefully, for this specific case.



The open circuit that has replaced C_2 isolates the voltage source from the rest of the circuit. It is clear that the marked current i is still $-I$ at this new equilibrium: $i(\infty) = -I$. The same argument holds for L_1, R_1 and C_1 having the same voltages and currents as at $t = 0^-$, in this circuit.

However, the voltage across C_2 is not the same as before. If we mark it with its positive side on the left, then KVL around the loop of R_2, C_2 and U, gives $u_{C_2}(\infty) = -IR_2 - U$.

2 Continuity

The current and voltage of an ideal resistor can change in no time at all: they can suddenly switch from a large positive to a large negative value, for instance. Seen as a function of time, such a step is a *discontinuity*.

The energy stored in a capacitor is related to its voltage, and the energy stored in an inductor is related to its current, as shown by the expressions $u^2C/2$ and $i^2L/2$ in Topic 06. To change these variables requires a change in stored energy, which in turn means a power flow during an interval of time. Significant time is needed in order to get significant energy in or out of a component.

This means that in a very short time-interval a capacitor cannot change its voltage significantly, and an inductor cannot change its current significantly, because these are the variables that are related to the components' energies. This principle is called *continuity*, indicating that these *continuous variables* can only change smoothly. They cannot step.

In an ideal circuit one could consider defining 'impulses' of energy, where finite energy arrives in vanishingly small time. When we consider for example a capacitor in parallel with a voltage source that has a step function, then the energy on the capacitor suddenly changes, and the current at that point is 'very big'. But as long as voltages and currents are required to be finite — or even quite strongly limited, as they tend to be in any real circuit that we are modelling — then the continuity principle is very reasonable over times that are very short compared to the times for new equilibria to be reached.

> Note, however, that the opposite way round is *not* true! The *current* through a capacitor, and the *voltage* across an inductor, *can* change instantaneously. They are linked to the *rate of change* of the continuous variables. (The continuous variables cannot have discontinuities in themselves, but they *can* have discontinuities in their time-derivatives.)

2.1 Principle of continuity calculation

The above description of continuity shows how to handle calculations of all the circuit quantities immediately *after* an equilibrium state has been disturbed by a change such as a switch or a component whose value has had a step-change. In other words, continuity allows us to solve our circuit at $t = 0^+$.

Let's define the time when the change happens as being $t = 0$. The equilibrium state that was present *before* the change can be found as described in the previous subsection. This gives the circuit quantities at $t = 0^-$; the only quantities that are needed from this state are the continuous variables of the reactive components, representing stored energy (memory). Then, at $t = 0^+$ we know that these continuous variables *are the same as before*: they have not had time to change.

We solve the circuit for $t = 0^+$ by using any new values of components or switches, *and* assuming that all the continuous variables are the same as at $t = 0^-$. In this way, we can find all the voltages and currents in the circuit, at this time $t = 0^+$.

For equilibrium calculations we replaced capacitors and inductors by open and short circuits, to make the diagram more easy to understand. A similar sort of simplification is possible for the calculation using continuity. Instead of zeroed current and voltage sources, the capacitors and inductors at $t = 0^+$ can be drawn as voltage and current sources with values that match the continuous variables. Some people may find it helps to redraw the circuit with these sources marked.

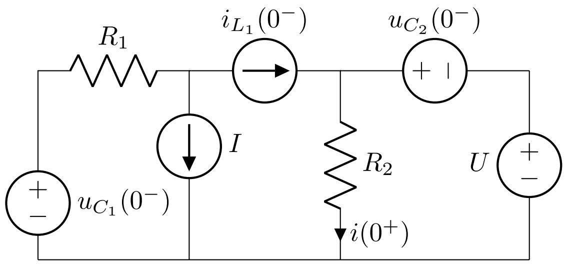
The following reasoning is how we can justify doing this. For a capacitor at $t = 0^+$, we have claimed that the voltage is definitely known, by continuity from its equilibrium value. However, its current and therefore its rate of change of voltage are *not* known; the current depends on how the rest of the circuit responds to the capacitor's voltage. This description matches exactly with a voltage source: its voltage is known, but its current is entirely determined by how the rest of the circuit responds to its voltage. For this reason, a voltage source can replace the capacitor. A dual description applies to replacing an inductor with a current source.

Beware that this identical behaviour is true *only at that instant* $t = 0^+$! The continuous variables may start to move towards new equilibrium conditions in the changed circuit, so we cannot assume they remain at their old equilibrium value. If the circuit causes a large values of the discontinous variables of current through a capacitor or voltage across an inductor, then the continuous variables with have a high rate of change.

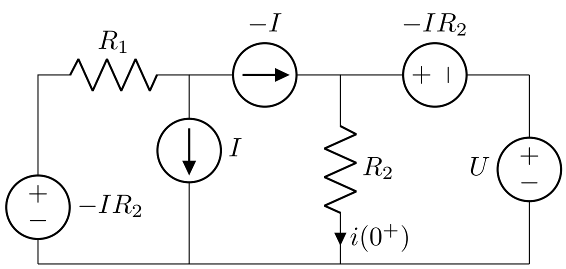
2.2 Example of continuity calculation

The same circuit is used here as in the equilibrium example. The following diagram shows the replacement of reactive components with sources that match their continuous variables, known from the equilibrium condition at $t = 0^-$.

The other components are given whatever values are appropriate for the time $t = 0^+$: only the voltage source is time-dependent in this circuit, and its value is U at this time.



The names like $i_{L_1}(0^-)$ are a little tedious to keep writing while solving. In our case, we found that these continuous variables in the equilibrium condition were given by quite short expressions, so we can just write these into the diagram. Be careful about the signs! They depend on which way you drew the sources that replaced the capacitors and inductors. Check this diagram, comparing to the equilibrium calculation.



If the expressions from the equilibrium condition had instead been long, it would probably have been better to have given them short names like U_x, U_y, I_z, then substituted the actual values at the end.

The above diagram is a classic dc circuit to solve; every component value is expressed in known quantities, and all connections are shown. All the circuit variables at $t = 0^+$ can therefore be found by dc analysis. A few examples are given in the following.

The marked current $i(t)$ at $t = 0^+$ can be found from KVL in the rightmost loop; $i(0^+) = \frac{U-IR_2}{R_2} = \frac{U}{R_2} - I$. This current depends on the actual voltage source U *and* on the voltage that C_2 was charged to at $t = 0^-$.

By KCL above R_2 it is then seen that a current of U/R_2 comes out of the + terminal of the source that represents C_2.

The voltage across the inductor can be found by KVL around the outer loop, where $IR_2 - U - IR_2 + 0 \cdot R_1 = u_{\mathrm{L}_1}(0^+)$. The inductor voltage is here defined with its positive reference where the source current arrow goes in. The current in R_1 is seen to be zero from KCL at the node above source I.

When these sorts of calculations become tough, and solutions aren't obvious by just one or two applications of KCL, KVL or Ohm's law, consider nodal analysis. Be careful and systematic. Double-check working: check dimensions at each step.

3 Summary/Advice

Existing dc methods can be used to find steady equilibria for circuits where no changes are happening to circuit quantities, and to find the circuit quantities when a sudden change has happened immediately after a time when the continuous variables are known.

The key is to remember rigidly which is the *continuous* variable for the capacitor and for the inductor. It is this *and only this* which cannot change instantaneously for that component.

A common mistake is to assume that the discontinuous variable will not change, or is zero. That's a bit like the mistake of assuming a current source has zero voltage, or a voltage source has zero current. Remember that each component can define just one of two 'degrees of freedom'.

Similarly, for equilibrium, think carefully about which variable must be constant: for example, a capacitor in equilibrium should not be charging, so we expect $i = 0$ and $\frac{\mathrm{d}u}{\mathrm{d}t} = 0$. By duality, the inductor can be seen to have $u = 0$ and $\frac{\mathrm{d}i}{\mathrm{d}t} = 0$ in equilibrium. Make the right choice of open- or short-circuit!

Plenty of clear, well marked diagrams are useful when handling these equilibrium and continuity calculations.

4 — Extra —

There's not an awful lot I can think of just now, that is relevant to this Topic but is not core material. If questions come up during the week (lecture, email, tutorial, hw) I will consider including answers in this section.

4.1 Time-stepping Equilibrium & Continuity

In the next Topic we will look at using differential equations to find solutions for circuit quantities as time-functions over all time, not just the special cases of equilibrium values and immediately after sudden changes, that we studied this time.

However, we could make an approximation of the time-functions using what we already have covered in this topic. Starting from a known state of the capacitor voltages and inductor currents, the currents in the capacitors and the voltages across the inductors can be found by dc analysis. Then, using the component values C and L, the time rate of change of the continuous variables can be found. The continuous variables a short time Δt later can be approximately found by adjusting their initial values by an amount $\Delta t \frac{\mathrm{d}\cdot}{\mathrm{d}t}$. For a capacitor C, for example, this could mean that its voltage u_x is

$$u_x(t+\Delta t) \simeq u_x(t) + \Delta t \, \frac{1}{C} \, i_x(t),$$

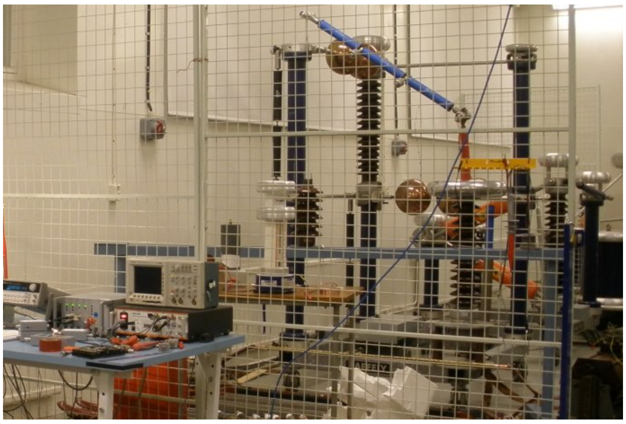
where i_x is the capacitor's current. This is only approximate: it assumes the rate of change throughout the interval Δt is constant, equal to its value at the start. Too large a choice of Δt would make this a very bad assumption. The equations (5) and (6) show the exact alternative to the above, using integrals $\int \mathrm{d}t$ instead a simple multiplication Δt.

By repeatedly making this calculation, setting new values of continuous variables for the sources representing the capacitors and inductors, then finding the other variables and calculating the change in the continuous variables, a complete time-solution can be approximated, without looking at analytic solutions of differential equations. This is what is done in numerical programs for solving such circuits, although with much more sophisticated algorithms for the numerical integration.

4.2 A later use of Equilibrium & Continuity

Other than that, here is an example of a circuit where one has a practical reason for wanting to find an equilibrium state. It's slightly amusing in that it's really a simple circuit compared to what we study in this course, and yet it leaves a lot of students at Masters level rather puzzled (because they've forgotten all about these sort of calculations, or never studied them properly, or don't see the connection, . . .).

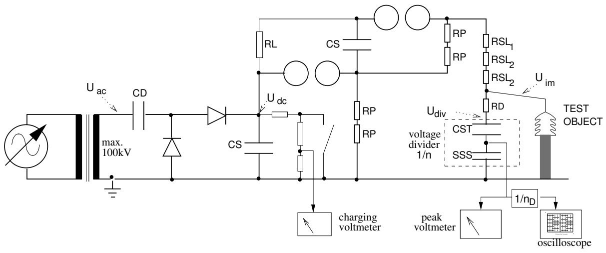
A lab task in the course High Voltage Engineering involves "lightning impulses" being generated by a construction that is poorly shown in this picture. I will take a better picture some day.



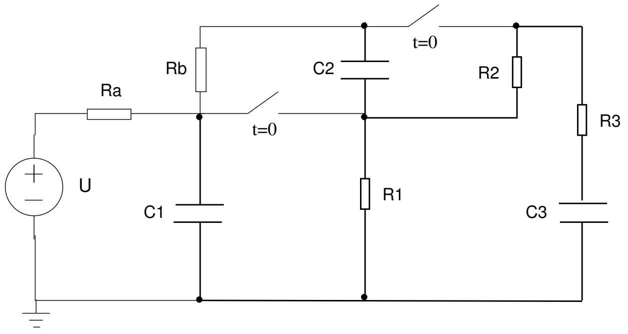
The output, to some test object, rises in about 1 µs from zero to a peak of somewhere between 30 kV and 300 kV, depending on the setting. Then the voltage falls more slowly, so that it's halved in about 40 µs.

Before a voltage impulse can be provided at the output, the two capacitors labelled `CS` in the following diagram have to charge up through resistors. Then sparks are made to happen in the highly stressed air between two pairs of copper spheres: the air gap thus changes from a very good approximation of an open circuit, into a quite good approximation of a short-circuit.

A diagram of the entire circuit is the following, but it is hard to follow as it contains even the details of how the variable ac supply is converted to dc, and how the dc and impulse voltages are measured.



A simplification of just the relevant parts uses switches to model the spark-gaps, and a dc Thevenin source to model the transformer, capacitor and diodes on the left of the circuit. It also combines some series-connected capacitors or resistors.



A common question to ask is what the potentials are at various points in the circuit when the source has value U, and the spark gaps have *not* broken down (they are open circuit), and the circuit has reached equilibrium. Then the next question is what happens just after this if both switches close (i.e. if both spark-gaps fire). For example, what will the potential of the node on the top of R_2 be just after the switches close?

Most of the MSc students can't answer that. I suspect they *could* if they just had it firmly fixed in their minds that capacitors look like open circuits in equilibrium, resistors with zero current have zero voltage, and capacitors don't instantaneously change their voltage. You might be doing this lab in a few years.

If you want a *really* challenging thing to think of, requiring that you at least understand what 'ac' is (even if not needing to know ac analysis methods) consider how the circuit on the left in the first diagram produces a dc voltage marked U_{dc} of *twice* the peak value of the ac voltage supplied from the transformer at the left (the heavy vertical lines). The circuit to analyse is just the two diodes and the capacitor `CD`: these form a simple [Voltage-Multiplier]. The later parts of the circuit with the sphere gaps are a simple [Marx-Generator] for generating an impulse (rising and falling) from the dc. The impulse can reach nearly twice the dc voltage, and therefore nearly four times the peak ac input voltage.

Bigger generators — like the one we used to have in what is now Operahögskolan (!) — can have dozens of capacitors and gaps, and occupy a tall building … like Operahögskolan. So let's fill some remaining space with a historic picture of the 2 MV Marx generator in that building, back in the year 2001.