Electric Circuit Analysis, KTH EI1120

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Topic 08: Time-functions

In the previous two Topics we introduced:

> reactive components — capacitor and inductor,
>
> the unit-step function and time-dependent switch, as ways of making abrupt changes in a circuit,
>
> equilibrium calculation in a circuit where all components have constant values,
>
> the continuity principle for calculation of circuit quantities immediately after a change of components or connections disturbs a known state.

Here in Topic 08 the main skill to develop is how to find a circuit quantity *as a function of time*, during some period instead of just at one point. This is a more difficult task, so we limit ourselves in this course to the following situation:

> During the time-period that we are studying, the circuit connections and component values are constant.
>
> There is only one independent capacitor or inductor affecting the part of the circuit where we are making this calculation, so we only have to deal with a first-order differential equation.
>
> We know the "initial conditions" of the circuit: that means that we know the currents in all inductors, and voltages on all capacitors, at the start of the time-period that we are studying.

In this type of problem, the time-period is often $t > 0$, meaning all time after the point that we define as the initial time $t = 0$.

The last of the three conditions above is often fulfilled *not* by our being told the initial values, but by our having to *calculate* them using equilibrium and continuity, given the following information:

> A sudden change happens in the circuit: for example, a switch changes the connections of the components, or a component's value is changed.
>
> Before the change, equilibrium is assumed, and therefore the required states of capacitors and inductors can be calculated.
>
> The time-period we are studying starts immediately after the change, so we can use the principle of continuity to find the initial conditions for our differential equation solution.

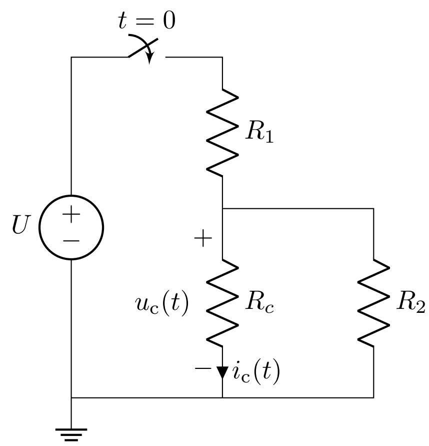
In the following sections, several methods are shown for finding time functions for circuits that have one reactive component.[1]

Please note the following!
The "circuit quantity" for which a time function is found could be *any* of the voltages or currents in the circuit. It *is* quite common, in exam questions and real applications with a single reactive component, that one wants to find the time-function for the voltage or current in the reactive component.
But if a voltage or current somewhere else in the circuit is sought, then a convenient method of solution is to find the time-function of the reactive component's continuous variable, and then to use this known value on the reactive component to calculate other quantities in the circuit.
This can be done by the same approach as was used in continuity calculations: the capacitor or inductor can be treated as a voltage or current source, whose value is the time-function that has already been found. If all the other components are sources and resistors, then an algebraic (not differential) equation can be found, to calculate other quantities in terms of this time-function for any moment in time.

1 Differential equation for the circuit

1.1 Practice *without* C or L

Suppose that we have the following circuit: notice that is does *not* contain reactive components.



Without any reactive components, this is a static circuit, where the solution at any time depends only on the components' values at that time. So let's find what is the marked voltage u_c at any time after zero.

[1] In fact, this ability is a bit more general: we could have said "finding time functions for circuits that have one relevant, independent reactive component". Irrelevant reactive components would be those that are isolated from the sought quantity by for example being in a series branch with a current source, or hidden behind a voltage source or short or open circuit. Non-independent reactive components would be for example two inductors, or two capacitors, that are in series or parallel connection and can therefore be reduced to a single component.

(Clearly, $u_c(t) = 0$ for $t < 0$, since the set of resistors has no source connected to it before the switch closes, and resistors do not store energy.)

One way we could think of solving for u_c is to consider KCL in the node that joins the resistors. This node has potential u_c, so we see that

$$\frac{U - u_c}{R_1} - \frac{u_c}{R_2} = i_c.$$

That doesn't look good, because now there are two unknowns, u_c and i_c, in the one equation. But the resistor is described by Ohm's law, which gives us a further equation, $i_c = \frac{u_c}{R_c}$. Substituting this, we get an equation with only one unknown, u_c,

$$\frac{U - u_c}{R_1} - \frac{u_c}{R_2} = \frac{u_c}{R_c}$$

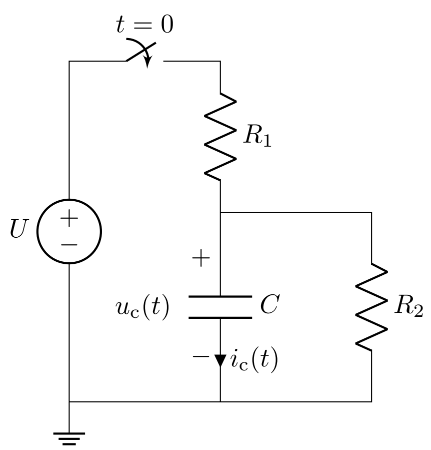
which can be solved,

$$u_c = \frac{UR_2R_3}{R_1R_2 + R_1R_c + R_2R_c}.$$

1.2 Try again *with* a capacitor

The point of the above rather strange way of solving a simple resistor circuit was to show how similar the procedure can be when one of the components is a reactive component. You really only need your old knowledge of dc circuits, and knowledge of the equations for i and u in a capacitor or inductor, to be able to write the differential equations for a circuit with one of these reactive components; then you can use your differential-equation skills to handle the equation!

The resistor R_c is now replaced with a capacitor C.



Here, it is still true that

$$\frac{U - u_c(t)}{R_1} - \frac{u_c(t)}{R_2} = i_c(t),$$

but in order to get the equation to have only one unknown variable, we have to use the capacitor's circuit equation, $i_c(t) = C\frac{\mathrm{d}u_c(t)}{\mathrm{d}t}$, which can be substituted to give

$$\frac{U - u_c(t)}{R_1} - \frac{u_c(t)}{R_2} = C\frac{\mathrm{d}u_c(t)}{\mathrm{d}t}.$$

Rearranged to a more standard form,

$$\frac{\mathrm{d}}{\mathrm{d}t}u_c(t) + \frac{1}{\frac{R_1R_2}{R_1+R_2}C}u_c(t) = \frac{U}{R_1C},$$

which corresponds to $\frac{\mathrm{d}y(t)}{\mathrm{d}t} + ay(t) = b$, whose solution is $y(t) = \frac{b}{a} + k\mathrm{e}^{-at}$, with k an arbitrary constant.

The general solution for the capacitor's voltage is therefore

$$u_c(t) = \frac{UR_2}{R_1 + R_2} + k\mathrm{e}^{-t/\tau}$$

where $\tau = \frac{R_1R_2}{R_1+R_2}C$. But we need to know the *initial condition* (sv: *begynnelsevärde*)[2] in order to find k.

In this circuit, we can assume the circuit was in equilibrium at time $t = 0^-$, as it has been the same "for all time" and doesn't have any problems such as capacitors in series with ideal current sources.

In this case, the capacitor must have had zero voltage at $t = 0^-$. That can be seen as it is in parallel with a resistor that would carry no current: the only paths through R_2 would be through the capacitor, which behaves as an open circuit in equilibrium, or through the switch which is open. The voltage on a capacitor is continuous, so we can assume the same state immediately after the switch closes: $u_c(0^+) = 0$. Putting in this condition, and noting that $\mathrm{e}^0 = 1$,

$$u_c(0^+) = 0 = \frac{UR_2}{R_1 + R_2} + k$$

from which we see $k = -\frac{UR_2}{R_1+R_2}$. The simplified solution for all $t > 0$ is then

$$u_c(t) = \frac{UR_2}{R_1 + R_2}\left(1 - \mathrm{e}^{-t/\frac{R_1R_2}{R_1+R_2}C}\right).$$

which could even be extended to be valid into negative times, by multiplying it with a unit step at $t = 0$!
A neater way of writing the above is

$$u_c(t) = U_f\left(1 - \mathrm{e}^{-t/\tau}\right),$$

where the values of U_f and τ can be seen by comparison with the earlier expression. The significance of these two values is that U_f is the "final" value of $u_c(t)$ as $t \to \infty$, and τ is a time-constant (see later section) showing how fast the change happens.

See how the final value in this case looks like the voltage-divider equation: this is understandable from the circuit diagram if we replace the capacitor with an open circuit to calculate the final equilibrium state.

[2] Or, at least, the state of energy on the capacitor at *some* point in time, even if not necessarily the earliest point.

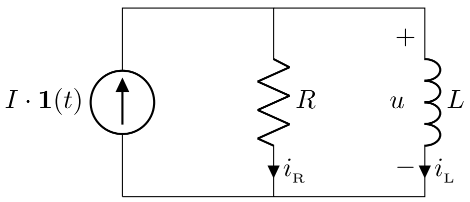
We see that this is a *final* value by considering that $\left(1 - \mathrm{e}^{-t/\tau}\right)$ becomes 1 when $t \to \infty$.

In other transients solutions there could be instead an initial value that is relevant. For example, try solving this same circuit for the current $i_\mathrm{c}(t)$ instead of the voltage $u_\mathrm{c}(t)$. Or find this current by differentiating the voltage that we already found: $i_\mathrm{c}(t) = C\frac{\mathrm{d}}{\mathrm{d}t}u_\mathrm{c}(t)$. The current has an initial non-zero value at $t = 0$, then falls to zero towards long times.

In still other cases, the initial and final values could both be non-zero: this would have happened in the above case if the initial condition had been non-zero, which could have been achieved by having a further resistor in parallel with the switch.

1.3 Another example

Here is another example, with two ways of solving it. It's interesting to note how in one case the I term comes in from the arbitrary constant, and in the other it remains present in the equation.



1.3.1 One method

At $t > 0$,

$i_\mathrm{L} = I - i_\mathrm{R}$ (KCL)

$i_\mathrm{R} = u/R$ (Ohm)

$\therefore \quad i_\mathrm{L} = I - \frac{u}{R}$ (the above two)

$u(t) = L\frac{\mathrm{d}}{\mathrm{d}t}i_\mathrm{L}(t)$ (relation of i and u in an inductor)

$\therefore \quad u(t) = L\frac{\mathrm{d}}{\mathrm{d}t}\left(I - \frac{u(t)}{R}\right) = -L\frac{\mathrm{d}}{\mathrm{d}t}\frac{u(t)}{R}$

simplify: $\frac{\mathrm{d}}{\mathrm{d}t}u(t) = \frac{-R}{L}u(t)$.

This is a "homogeneous" ODE, i.e. there is no constant term: so b=0 in the standard form $\frac{\mathrm{d}y(t)}{\mathrm{d}t} + ay(t) = b$. The solution is $u(t) = k\mathrm{e}^{-tR/L}$, where k needs to be found.

The initial voltage, $u(0^+)$, is IR.
Why? There is no current in the inductor at this time, $i_\mathrm{L}(0^+) = 0$ (explained in the next method); therefore, all the current I must pass through R at the start, until the resulting voltage has had time to increase the inductor's current. By Ohm's law, $u(0^+) = (I - i_\mathrm{L}(0^+))R = IR$.

Now write the solution of $u(t)$, putting in the known values for when $t = 0^+$:
$u(0^+) = IR = k\mathrm{e}^0 = k$, so $k = IR$.

The complete solution is then $u(t) = IR\,\mathrm{e}^{-tR/L}$.

We can check for reasonableness, for example by considering $t \to \infty$, in which case there should be an equilibrium and the inductor is therefore a short-circuit, i.e. zero voltage. This is true: $\mathrm{e}^{-tR/L} \to 0$ as $t \to \infty$. The dimensions are also consistent.

1.3.2 Another method

Another approach is to start by solving for the inductor's current $i_\mathrm{L}(t)$, at $t > 0$, which is the continuous variable in the circuit. This solution can then be differentiated to find $u(t)$ from $L\frac{\mathrm{d}i}{\mathrm{d}t}$.

The difference in how we start is that we eliminate the $u(t)$ instead of the $i_\mathrm{L}(t)$ in the KCL equation.

$u(t) = L\frac{\mathrm{d}}{\mathrm{d}t}i_\mathrm{L}(t)$ (i and u in inductor)

$u(t) = Ri_\mathrm{R}(t)$ (Ohm's law)

$\therefore \quad Ri_\mathrm{R}(t) = L\frac{\mathrm{d}}{\mathrm{d}t}i_\mathrm{L}(t)$

$I = i_\mathrm{L}(t) + i_\mathrm{R}(t)$ (KCL)

$\therefore \quad I - i_\mathrm{L}(t) = \frac{L}{R}\frac{\mathrm{d}}{\mathrm{d}t}i_\mathrm{L}(t)$

and this can be written in the general form of

$\frac{\mathrm{d}}{\mathrm{d}t}i_\mathrm{L}(t) + \frac{1}{L/R}i_\mathrm{L}(t) = \frac{1}{L/R}I$,

which by the solution $\left\{\frac{\mathrm{d}}{\mathrm{d}t}y(t) + ay(t) = b\right\} \to \left\{y(t) = \frac{b}{a} + k\mathrm{e}^{-at}\right\}$ gives

$i_\mathrm{L}(t) = I + k\mathrm{e}^{-t/(L/R)}$

The ratio L/R is a time-constant, similarly to the product CR that we saw in an earlier example. So we can write the exponent as $-tR/L$ or $-t/(L/R)$, or as $-t/\tau$ where $\tau = L/R$.

The equilibrium state of the inductor current before the current source switched from 0 to I must have been $i_\mathrm{L}(0^-) = 0$. That is because any current in the inductor would have to pass through the resistor; it would therefore generate a voltage across the resistor, which would also be the voltage across the inductor as they are in parallel; this voltage would cause a changing current ($\frac{\mathrm{d}}{\mathrm{d}t}$) in the inductor, pushing it towards zero.
Alternatively: any current in the resistor means power, and over time this means energy; the inductor can only start with a finite energy, so the current cannot continue indefinitely.

Therefore, by continuity, $i_\mathrm{L}(0^+) = i_\mathrm{L}(0^-) = 0$. Putting in this condition at time $t = 0^+$,

$i_\mathrm{L}(0^+) = 0 = I + k\mathrm{e}^{-0/(L/R)}$

$\therefore \quad 0 = I + k$, meaning that $k = -I$.

The full solution is then
$i_\mathrm{L}(t) = I\left(1 - \mathrm{e}^{-t/(L/R)}\right)$.

That is one classic form of solution of a first-order LR or CR circuit, going from zero to a final value. It seems intuitively right for an inductor that started with zero current: the initial current is zero, but in the final equilibrium the inductor looks like a short circuit, so all the current I passes through it instead of through the resistor.

However … the original task was to find the voltage $u(t)$.
This is $u(t) = L\frac{\mathrm{d}}{\mathrm{d}t} i_{\mathrm{L}}(t)$.
Differentiating the above solution for $i_{\mathrm{L}}(t)$,

$u(t) = L\frac{\mathrm{d}}{\mathrm{d}t}\left[I\left(1 - \mathrm{e}^{-t/(L/R)}\right)\right] = LI\frac{R}{L}\mathrm{e}^{-t/(L/R)}$.

$u(t) = IR\,\mathrm{e}^{-t/\tau}$.

That looks like what we got from the first method in Section 1.3.1.

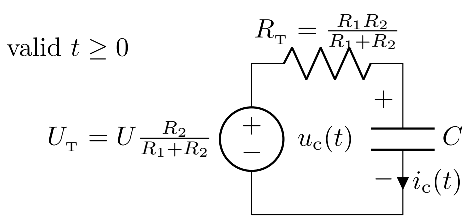
2 Equivalent Source method

This is another way of coming to the solution above. The two-terminal equivalent concept (Topic 04) is that any piece of circuit that connects to other things with just two terminals (nodes) can be represented by a voltage or current source and a resistor. This applied to the components we knew about then. It does *not* generalise to reducing a circuit with capacitors and inductors into a simple equivalent for transient solutions. On the other hand, we will see in Part C of the course that it does apply to the situation of steady-state sinusoidal analysis.

However, when we only have one or a few reactive components, in one part of a circuit, it may be helpful to reduce the rest of the circuit (independent and dependent sources, and resistors) to its Thevenin or Norton equivalent. That is a valid move: the components in the equivalent are all static ones. The advantage of doing this is that the total circuit then becomes just a single source and resistor, with one or a few reactive components connected to them.

In our case, we've said that our exams extend only to circuits with a single relevant capacitor or inductor. Then we only need to learn how to solve the case of each of these components connected to a Thevenin or Norton source, with some initial condition that we find from equilibrium. We already know how to find this equivalent source for the circuit other than the reactive component (Topic 04).

Applying the above method to the example used in the earlier section, we see that the capacitor and the Thevenin equivalent of all the other components can be written as in the following diagram. This diagram is only valid for $t \geq 0$; before that time, the switch is open so the equivalent seen by the capacitor is just R_2.



From this, the time-constant and initial and final value that were found in the previous section can be identified easily,

> The initial value $u_{\mathrm{c}}(0^+)$ is the same as $u_{\mathrm{c}}(0^-)$, by continuity. We could just as well call this $u_{\mathrm{c}}(0)$, since continuity means there is no significant difference in the value between these very close time-points. The equivalent circuit was only drawn for $t \geq 0$, since the different position of the switch results in a different circuit for times $t < 0$. It is therefore necessary to look back to the original circuit to find the initial equilibrium, $u_{\mathrm{c}}(0^-) = 0$, from which continuity tells us that $u_{\mathrm{c}}(0^+) = 0$.
>
> The final value, when $t \to \infty$, can be easily treated by equilibrium in the equivalent circuit: it is equal to the Thevening source value, $u_{\mathrm{c}}(\infty) = U_{\mathrm{T}} = U\frac{R_2}{R_1+R_2}$.
>
> The circuit's time-constant is $\tau = R_{\mathrm{T}}C$, therefore $\tau = \frac{R_1R_2C}{R_1+R_2}$.

We simply have to find a function that starts at the initial value, and moves asymptotically towards the final value, with its rate of change determined by the time-constant. In general, if the initial and final values are u_{i} and u_{f}, this function will be

$$u(t) = u_{\mathrm{f}} + (u_{\mathrm{i}} - u_{\mathrm{f}})\mathrm{e}^{-t/\tau},$$

which follows from $\mathrm{e}^{-0} = 1$, and from $\mathrm{e}^{-t} \to 0$ when $t \to \infty$.

In our specific example, above, we have $u_{\mathrm{i}} = 0$, so the solution is just $u_{\mathrm{f}}\left(1 - \mathrm{e}^{-t/\tau}\right)$. In many cases, $u_{\mathrm{f}} = 0$, which gives a simple decay, $u_{\mathrm{i}}\mathrm{e}^{-t/\tau}$.

Depending on your taste, you may prefer to use this method for every time-function solution in this course (with a single capacitor or inductor in a circuit), by converting the rest of the circuit to your favourite choice of a Thevenin or Norton equivalent. Or, you might prefer just to develop the differential equation from the circuit equations, and solve it directly, as in the previous section. There will not be a requirement to use a specific method for this transients section of the course, so you are free to choose.

The equivalent-circuit method has *some* advantage by encouraging the "intuitive" understanding that all of these first-order circuits have a particular shape of the response: one can quickly see the result without even thinking about the differential equations and formal solutions. But as soon as the circuit has more than one reactive component (that is relevant and can't be simplified to a single component), we have to go back to writing the equations based on normal circuit analysis, and solving the resulting differential equations.

3 Time-constants

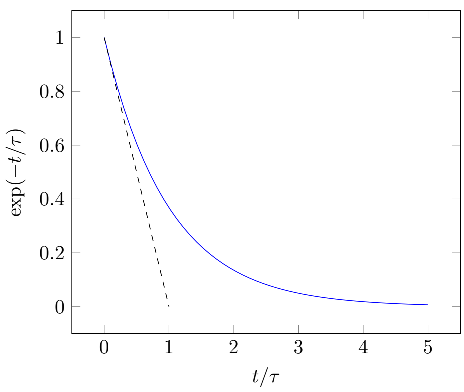
Linear first-order differential equations, like the ones we can get with a single reactive component in a linear circuit of resistors and sources, come up in many situations: radioactive decay, heat transfer, electricity, ...

The reason is generally that the change in "something" depends on how much of that something there still is! For example, a capacitor discharging through a resistor has a discharge current proportional to its voltage (Ohm's law), and this voltage is proportional to the charge remaining on the capacitor. Hence, the rate of change of charge (the current) is proportional to the amount of charge, at any time.

In these situations, the value that we called a in the function e^{-at} has dimensions of $[s^{-1}]$, and is sometimes replaced with a time $\tau = \frac{1}{a}$ called the time-constant, so that the function is written $e^{-t/\tau}$.

When $t = \tau$ the result is e^{-1} which is about 0.37. Sometimes people define a time-constant in just that way: *the time for the function to have only 37% remaining of the difference between its initial and final values.*

Another way to see it is: *the time it would take to reach the final value* if *the inital rate of change were to stay constant"*.



The figure (above) shows an exponential function $e^{-t/\tau}$ at $t = 0$, together with its tangent (dashed) at $t = 0$. The tangent has a gradient of $-\frac{1}{\tau}$, which can be calculated from $\frac{d}{dt}e^{-t/\tau}$ at $t = 0$. It therefore reaches zero at time $t = \tau$. In contrast, the exponential slows down more and more as it approaches its final value.

4 Stability

In earlier Topics we discussed the assumption that a constant equilibrium value of every variable is reached in a circuit that has constant sources. Some of the necessary conditions were that ideal sources should not be connected in a way that keeps changing the continuous variable, e.g. a capacitor connected directly across a current source, or an inductor across a voltage source.

This can be seen more clearly, for the case of a single reactive component, by using the Equivalent Source method to replace all the rest of the circuit with a Thevenin or Norton source. If the equivalent source has a positive resistance, the circuit will be stable[3] As we have seen from earlier examples of calculating time-functions that have terms $e^{-t/CR}$, a positive resistance leads to a decaying exponential term that tends to zero. If instead the source has zero or negative resistance, you should not assume stability. As already mentioned, realistic power-oriented circuits without special control systems would naturally have some positive resistance.

5 Examples of first-order circuits

This Section considers all the possibilities of capacitors and inductors connected to Thevenin or Norton sources.

Remember that this has a broad relevance, since all circuits that are just sources and resistors and *one* reactive component can be replaced with a Thevenin or Norton source connected to that reactive C or L component.

(Please do not think of this as something you should memorise! It's probably better to learn the *method* of solving, by trying to solve these examples before checking the answers.)

There are several possibilities:

Thevenin *or* Norton source,

Capacitor *or* Inductor component,

solve for Voltage *or* Current in the component.

That's eight possibilites already. It could be more if we considered solving for another variable like the voltage across a Thevenin-source resistance, or the current through a Norton-source resistance.

We could add another eight by including as separate cases the ideal voltage and current sources, where there is no resistor. These are really just limiting cases of the above, where the Thevenin resistance or Norton conductance tends to zero. They give one of:
* Trivial results, like finding the current in series with a constant current source.
* Delta-function results (big pulse), like current into a capacitor when a parallel voltage-source has a step.
* Constantly changing, like voltage on a capacitor connected to just a current source.

Therefore, we focus on the cases of a Thevenin or Norton source connected to a C or L component.

We should be able to halve the number of different forms of equation by seeing that the result should be

[3] Or at least, it will be stable if the capacitance or inductance is positive – and yes, you *could* emulate negative ones using dependent sources or opamps!

the same for a Thevenin or Norton equivalent (even if the steps we do in order to *find* the result are *not* the same).

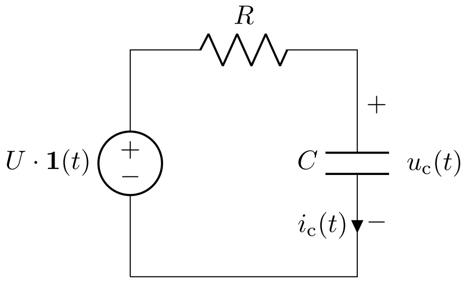
Then we should further be able to halve the number of different cases by thinking of duality: for example,
"the current in an inductor when connected to a Norton source with a step at $t = 0$"
should have the same equation as
"the voltage on a capacitor when connected to a Thevenin source with a step at $t = 0$".
In order to make the dual equations have exactly the same form, one would also need to swap resistance and conductance.

The solutions for voltage and current are shown in the diagram below, for each case.

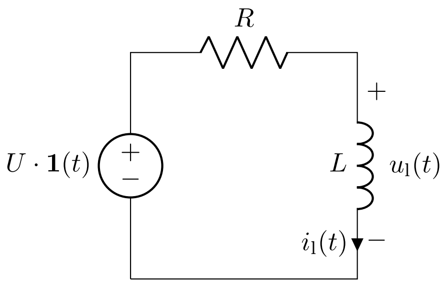
Note that in the shown situations, the initial condition at $t = 0^-$ is known to be zero. That is because we can see that the reactive component has been connected directly across a resistor for all the previous time, due to the source being zero at $t < 0$. Any energy in the capacitor or inductor that might have been in the reactive component a long time ago would have been consumed in the resistor.

You can reason this by saying e.g. that any charge on the capacitor means a voltage, and voltage means a current through the resistor, and the current means charge moving out of the capacitor (check the direction); or current in the inductor means current through the connected resistor, which means voltage across the resistor, which means rate of change of inductor current.

Start with a Thevenin source.

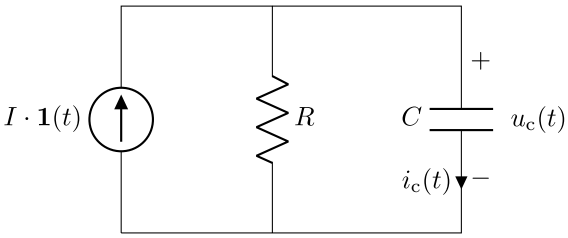


$$t > 0 \begin{cases} u_c(t) = U\left(1 - e^{-\frac{t}{CR}}\right) \\ i_c(t) = \dfrac{U}{R}\, e^{-\frac{t}{CR}} \end{cases}$$

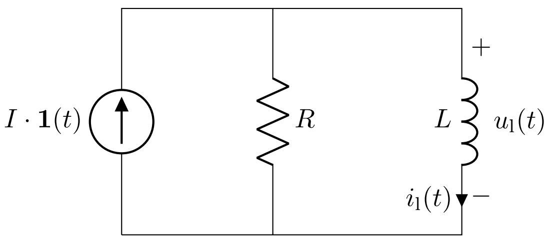


$$t > 0 \begin{cases} u_l(t) = U\, e^{-\frac{t}{L/R}} \\ i_l(t) = \dfrac{U}{R}\left(1 - e^{-\frac{t}{L/R}}\right) \end{cases}$$

Now the Norton equivalents. These give the same basic equations, but with substitutions based on $IR = U$, as would be expected from the principle of source-transposition.

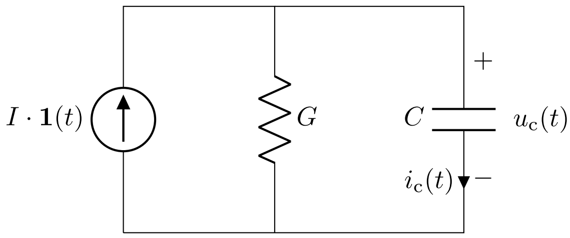


$$t > 0 \begin{cases} u_c(t) = IR\left(1 - e^{-\frac{t}{CR}}\right) \\ i_c(t) = I\, e^{-\frac{t}{CR}} \end{cases}$$

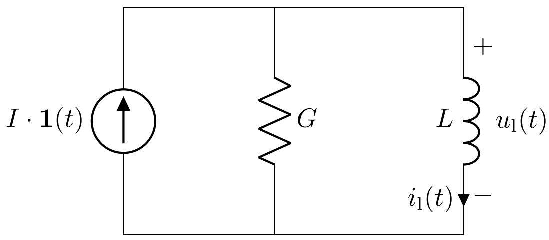


$$t > 0 \begin{cases} u_l(t) = IR\, e^{-\frac{t}{L/R}} \\ i_l(t) = I\,\left(1 - e^{-\frac{t}{L/R}}\right) \end{cases}$$

These Norton equivalents can be rewritten to use $G = 1/R$ instead of R, giving the full "dual" where the equations have exactly the same form as the dual cases with Thevenin sources. A dual reasoning could be followed all the way through the solution. This should, I hope, appeal to those people who like patterns.



$$t > 0 \begin{cases} i_c(t) = I\, e^{-\frac{t}{C/G}} \\ u_c(t) = \dfrac{I}{G}\left(1 - e^{-\frac{t}{C/G}}\right) \end{cases}$$



$$t > 0 \begin{cases} i_l(t) = I\,\left(1 - e^{-\frac{t}{LG}}\right) \\ u_l(t) = \dfrac{I}{G}\, e^{-\frac{t}{LG}} \end{cases}$$

It could be educational to make a full derivation by the direct ODE method, of each of the preceding cases. One hint is to find the continuous variable, then simply differentiate that expression (and multiply by the component value C or L) to find the other variable.

Please note that the above equations for the simple three-component circuits are just special cases with *zero initial condition* of the continuous variable. In the general cases $y(t) = y(\infty) + (y(0) - y(\infty))\, e^{-t/\tau}$,

6 — Extra —

6.1 Time-stepping solution

A shorter version of this was also in the Topic 07 'Extra'. At any instant, the only thing we need to know about the capacitor or inductor, in order to be able to solve for the voltages and currents in the circuit, is the energy-related (continuous) variable.

On the other hand, to solve for the rates of change (time-derivatives) of these continuous variables, which describe how the stored energy changes over time, the other variables and the values C and L of the reactive components are needed. That is clear from the circuit equations of capacitors and inductors, respectively $\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \frac{i(t)}{C}$, or $\frac{\mathrm{d}i(t)}{\mathrm{d}t} = \frac{u(t)}{L}$.

If we were happy with numerical approximation – and we generally are very happy with it, in practice! – then the following principle could be followed to find the time-development of all the variables in a circuit with multiple L and C components.

0) Start with known "states" (continuous variables), i.e. known voltages on capacitors and known currents in inductors. This information could come from an equilibrium calculation, or it could instead come from a measurement or assumption.

1) Using these states, and knowledge of the state of switches and time-dependent components at this first time-point, solve the circuit by the usual methods from dc analysis, to find whatever variables we are interested in, *and* the non-continuous variables on the reactive components: current in capacitors, and voltage across inductors.

2) Then we select a small time-interval ("timestep") δt, chosen to be short enough that "not much change" is expected in any state during this step. Clearly, the choice depends on the circuit being solved, and could be $< 1\,\mathrm{ns}$ or $> 100\,\mathrm{s}$ for different situations such as connections on a computer board, or a measurement system for insulation materials. From the relation $\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \frac{i(t)}{C}$ or $\frac{\mathrm{d}i(t)}{\mathrm{d}t} = \frac{u(t)}{L}$, the calculated value of the non-continuous variable can be used to estimate the continuous variable after the time-step, i.e. $i(t+\delta t) \simeq i(t) + \frac{1}{L}u(t)\delta t$.

3) At that next time-point, where the continuous variables have been estimated, keep going back to part '1)', to find the new rates of change.

The above list is *sort of* what a numerical solver will do in the programs that are used to solve transients in complicated (and possibly nonlinear) circuits. One complexity is the way of reducing that circuit equations to a nice form for the calculations. Another is that the actual equations for time-stepping integration will be more sophisticated than the above; you might already know about "implicit integration methods"?

Another option when there are several reactive components is to develop higher-order differential equations, then to solve these analytically or in a general-purpose numerical solver such as Mathematica's `NDSolve[]` (numerical differential equation solver).

Something else we haven't considered, but that has very great practical relevance, is when components do not all have constant values apart from a few discrete steps. For example, a source might have a time-function that is a triangle-wave (smoothly up, then smoothly down, repeating) or a sinusoid. This would fit with no extra effort into the numerical solution method described above: at each time point t, one would use the actual value of each source according to the function that describes its value, e.g. $\sin(\omega t)$. In the analytical methods, this means solving for a non-constant forcing function.

6.2 Reminder: Integrating Factor for 1st-order inhomogeneous case

We've relied on the claim of how to solve

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) + ay(t) = b.$$

In case you care about *why*, consider that it's a classic case for an integrating factor. The following reminder might help you. (For EI1110 a 'reminder' of the calculus course you took this month is probably not needed; other programs take Circuits a year after Calculus.)

If you happen to multiply all terms by e^{at}, you get

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t}\mathrm{e}^{at} + ay(t)\mathrm{e}^{at} = b\mathrm{e}^{at}.$$

What is nice about that is that the left-hand side is equivalent to $\frac{\mathrm{d}}{\mathrm{d}t}\left(y(t)\mathrm{e}^{at}\right)$, by the product rule: $\frac{\mathrm{d}}{\mathrm{d}t}(fg) = f\frac{\mathrm{d}g}{\mathrm{d}t} + \frac{\mathrm{d}f}{\mathrm{d}t}g$.

So the whole equation has the form $\frac{\mathrm{d}}{\mathrm{d}t}X(t) = b\mathrm{e}^{at}$. This has the solution $X(t) = \frac{b}{a}\mathrm{e}^{at} + k$, as can be found by integrating.

But we also know from earlier that $X(t)$ in our case is $\mathrm{e}^{at}y(t)$. Subtituting this into the above solution gives $\mathrm{e}^{at}y(t) = \frac{b}{a}\mathrm{e}^{at} + k$.

Here, k is needed because any additive constant will not affect the derivative, so the function is not fully determined until k is known.

Dividing all terms by e^{at} (multiplying by e^{-at}) gives, as earlier claimed,

$$y(t) = \frac{b}{a} + k\mathrm{e}^{-at}.$$

Even in the first of the above circuit-solution methods, we could have used this equation: in that case $b = 0$, so the result would immediately have been $y(t) = k\mathrm{e}^{-at}$.