Electric Circuit Analysis, KTH EI1120
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Topic 09: AC analysis

There is a lot of new stuff here. You're not expected to follow it all easily. The essential part for now is to be confident and well-practised about what to do to solve a simple circuit with one or two ac sources of the same frequency. Other parts will be built on later. But try to get up to speed on as much as you can before next time: it's all a useful basis for later Topics.

After only a couple of weeks spent on Transients, here we are at the start of our Part C: circuit analysis with *alternating current* (sv: *växelström*). The material in this Topic is mainly a foundation for the later ones.

We convert between sinusoidal time-functions (e.g. voltage and current) and the complex numbers that are used to represent these in ac analysis. We convert resistors, capacitors and inductors to complex numbers that describe the relation between magnitude and angle of the complex voltage and current.

Some useful rules for using complex numbers are described, and the complex-number method is used to solve some simple ac circuits. This Topic certainly has rather a lot of new concepts. The Exercises and Homework provide good examples of using these.

Please bear in mind that ac circuits entail all the ideas and methods of dc circuits, except that many of the quantities are complex instead of real. This thought can help to reassure us that all the principles of superposition, nodal analysis, irrelevance, equivalents, dependent sources, etc, can be applied to ac circuits. The algebra is the main thing making ac circuits seem more awkward; power calculations are also less simple, although this is partly a consequence of the complex numbers.

Main skills for this Topic:

Convert single or multiple R, L and C components to impedances or admittances.

Convert given "time-domain" voltages and currents to phasors.
E.g. $A\sin(\omega t+\alpha) \to A\angle\alpha$, etc.

Perform solution of the resulting ac circuit.
(Just like dc analysis, but with complex numbers!)

Convert solutions back to the time-domain.
E.g. $B\angle\beta \to B\sin(\omega t+\beta)$

Use superposition to handle sources of different frequencies.

Essential in the above:
Handling complex numbers!
$a+\mathrm{j}b$, $M\angle\phi$, $M\mathrm{e}^{\mathrm{j}\phi}$,

1 Rambling pseudophilosophical stuff

This course is defined as having three parts: dc, transient and ac. The transient part involved solving differential equations (although we also learned a way to cheat), and the ac part is going to require handling complex numbers (computers are the best way to cheat with this, but not in exams). Our ac analyses will involve the capacitor and inductor components that were introduced when we started with transients. But we will not now be using any *methods* from the transients subject: instead, all of the dc methods (nodal analysis, equivalent sources, superposition, opamps, etc) will again be important.

How do the dc, transient and ac situations differ, and why was this sequence chosen for the course? In terms of increasing *generality*, the sequence would be dc, ac, transient: dc is a special case of ac (where the frequency is zero), and both dc and ac are steady-state cases of the general time-domain solution, with excitation by constant or sinusoidal sources. Starting with dc is good because it is quite familiar from school, and does not require more maths than basic algebra with real numbers. Then, in transients and ac, a lot of solutions are based on the methods learned from dc circuits.

In power and communications, ac analysis is very widely useful, which is why we spend so much time on it. Putting it after transients allows us to believe more easily the fundamental claim that a sinusoidal source causes all circuit-quantities to be sinusoidal in a linear circuit when the steady state is reached: this can be inferred from the differential equations, and is 'confirmed' experimentally by the Lab 3 task. It is also hoped that, having begun to see how difficult it would be to solve a circuit with several capacitors and inductors by using differential equations directly, the ac approach will be seen as being actually a friendly and helpful alternative, in spite of the somewhat formidable complex-number equations that we will sometimes be seeing!

Reality is more thoroughly modelled as a field-problem in time: that implies partial differential equations in space and time, for all the electromagnetic phenomena. A circuit is a simplification by "discretising in space" so that we no longer need to care about the spatial coordinates, but only about connectivity and component values: with only time remaining as an independent variable, the circuit equations for a transient solution are ordinary differential equations. By making assumptions about particular types of steady state having been reached, we come to the other two cases: if all the [independent] sources driving the circuit are of constant value, then calculation of the equilibrium currents and voltages is a dc analysis; if instead the sources have sinusoidal values at one frequency, the equilibrium values of circuit quantities are sinusoidal too, and we can calculate them by ac analysis.

2 An ac solution example

A few years of experience with introducing ac circuits have suggested that it's best to start with an example, even if no one has any idea what it's all about. Then it's easier to explain and justify the later explanations of 'why'; you might also feel better to know in advance that the necessary steps are actually relatively simple. If you feel lost with the complex numbers, try Section 8 first.

2.1 The rules, with one source

Consider a linear circuit in these conditions:

> The circuit is driven by a single independent source.
>
> This source gives a sinusoidal function of time. For example, it might be a current source with a value $I(t) = \hat{I}\cos(2\pi f t)$, where $\hat{I}$ is the peak value of the sinusoidal current, and f is a frequency.
>
> Other than this source, the circuit can have any number of resistors, inductors, capacitors and dependent sources.
>
> The circuit is stable, so that after a large enough number of periods the transients from earlier changes have died away; this is the sinusoidal steady state, where all the circuit quantities (voltages and currents) are just the sinusoidal *forced response* caused by the independent source.
>
> We want to find one (or more) of the circuit quantities, and are only interested in the sinusoidal steady state value.

The limitation to a single independent source is only for simplifying this introduction. A few pages later in this Topic we will see how the ac method can be used for multiple independent sources. It is very easy if these sources all have the same frequency.

The limitation to sinusoidal steady state solutions is fundamental to ac analysis. Superposition can be used to combine the transient and steady response. Fourier methods and superposition can be used to handle periodic but non-sinusoidal sources. However, it is often true in practical power-related circuits that the steady state is closely reached within a few periods, and that the sources are "approximately sinusoidal".

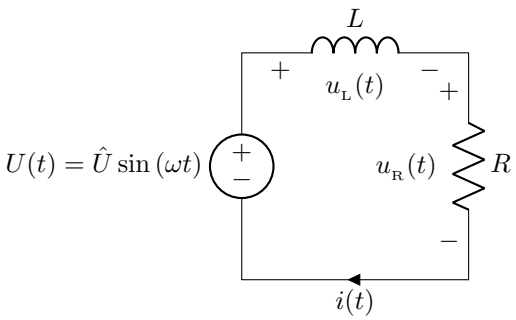
- Replace inductors L and capacitors C by *impedances*, respectively $\mathrm{j}\omega L$ and $1/(\mathrm{j}\omega C)$. The other symbols are the imaginary unit[1] $\mathrm{j} = \sqrt{-1}$, and the angular (radian) frequency $\omega = 2\pi f$.
- Replace the independent source's sinusoidal time-function with a *phasor*, which is a complex number indicating the amplitude and phase of a sinusoid at a particular frequency. There are several different common choices for defining the angle and scaling when converting between phasors and time-functions.
- Solve for the circuit quantity that you want. That's it ... it's just the same as you've done in dc circuits, except that some of the numbers are likely to be complex instead of real. That makes it a little more work. Your solution will be a phasor.
- If you wanted to find a time-function for the quantity you calculated, then you need to convert back from the phasor to a time-function, using the inverse procedure of the second step.

[1] In electrical applications it is common to use j instead of i as the imaginary unit, as we use i so much already for current (although we also like using j for current-density in fields calculations). Some people combine this unit with numbers after, and some before: e.g. j5 or 5j. In the former case the symbol might be called the imaginary operator.

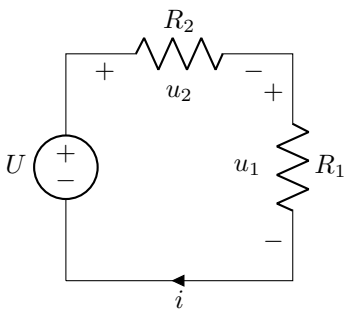
The circuit represented with time-functions and normal component values of capacitance, inductance and resistance, is said to be in the *time-domain*. The circuit when represented with complex numbers is said to be in the *frequency-domain*.

2.2 A simple application of the rules

"Find $i(t)$ in the following circuit!"
When there is such a question in the ac part of a course, we should just assume we want a steady-state ac solution.



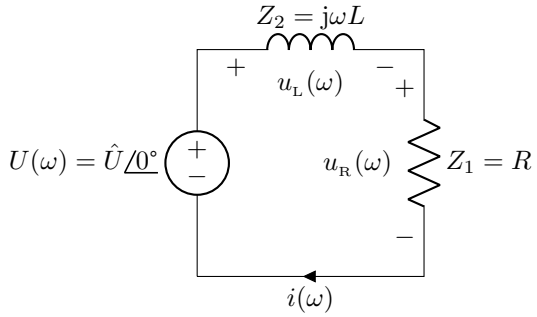
That might look a little worrying. What about the following dc circuit. That's not hard, is it?



The solution to that is clearly

$$i = \frac{U}{R_1 + R_2}.$$

The rules, listed above, said we solve the ac circuit like a dc circuit, after doing a few conversions of component values into complex numbers (impedance and phasors). Let's do the conversions. We'll define our phasors as having a magnitude equal to the peak of the sinusoidal time-signal, and a phase defined relative to the function $\sin(\omega t)$.



The notation $\hat{U}\angle 0^\circ$ means a complex number with magnitude $\hat{U}$ and angle of zero. We assume angles written as pure numbers are in radians, and angles in degrees are written with the $^\circ$ symbol. Slightly confusingly, the same symbol for angle can also mean "find the angle of this complex number": for example, $\phi = \angle a + \mathrm{j}b$.

The letter Z is used for impedance, which is a ratio of $\frac{\text{voltage}}{\text{current}}$ with more generality than resistance.

Writing quantities as functions of frequency[2] such as $i(\omega)$ is a useful reminder of whether we are thinking about time-functions or phasors. Otherwise it can sometimes be unclear: for example, a voltage of $U = 5\,\mathrm{V}$ could be a constant (dc) voltage in time, or a phasor representing a sinusoid at some assumed frequency, with size of U and angle of zero. Practical users of ac analysis seldom need to think about the time functions, and always work with phasors; then it's common to stop writing everything as a function of frequency, as we will do later in the course.

Now we can do that same calculation as in the dc circuit – compare this with the earlier expression:

$$i(\omega) = \frac{U(\omega)}{Z_1 + Z_2} = \frac{\hat{U}\angle 0}{R + \mathrm{j}\omega L}.$$

(It's not really necessary to say $\hat{U}\angle 0$; it's purely real, so it's the same number as $\hat{U}$.)

Now we have found the sought function $i(\omega)$. If we want it as a time-function we must convert back from a phasor to a time-function.

We know that every phasor in this circuit describes a sinusoidal time-function with frequency ω, in terms of peak magnitude and of phase relative to a sine function. That's what we chose when we converted the source to a phasor at the start. Suppose that some quantity $x(\omega)$ in the circuit has magnitude and phase of A and α, which can be written as $A = |x(\omega)|$ and $\alpha = \angle x(\omega)$. This will translate to a time-function

$$x(t) = A\sin(\omega t + \alpha).$$

We only need to find the magnitude and phase of the complex number $i(\omega)$, and put these into an equation like the above. The trouble is that $i(\omega)$ is a symbolic function, and the complex part is down in the denominator. We have to rearrange it to express the magnitude and angle separately.

You might already be able to show that

$$\frac{\hat{U}\angle 0}{R + \mathrm{j}\omega L} = \frac{\hat{U}(R - \mathrm{j}\omega L)}{R^2 + \omega^2 L^2} = \frac{\hat{U}}{\sqrt{R^2 + \omega^2 L^2}}\angle -\tan^{-1}\tfrac{\omega L}{R}.$$

Try Section 8 if you're lost with this.

From this expression, we see that

$$|i(\omega)| = \frac{\hat{U}}{\sqrt{R^2 + \omega^2 L^2}}, \qquad \angle i(\omega) = -\tan^{-1}\frac{\omega L}{R}$$

By putting these into the general expression for a time-function in our circuit, we get $i(t)$,

$$i(t) = \frac{\hat{U}}{\sqrt{R^2 + \omega^2 L^2}}\sin\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right)$$

That's the solution! As you gain familiarity, such solutions will come "pouring out of your pen".

3 Background to the rules

[Not complete yet. This section isn't part of what you *need*, but just what a few people might like to study in order to understand the background better.]

3.1 Intuitively

An example has already been shown in which sinusoidal time-functions are represented by complex numbers. All the quantities in the circuit are assumed to be sinusoidal, at some particular frequency ω.

It is sufficient to think of the complex numbers as representing the amplitude and phase of a sinusoidal time-function. However, there are various other ways to think of phasors.

Some people like to think of the phasor as a line in the complex plane, then to think of these lines (or the whole plane?) whizzing around (anticlockwise) at the angular frequency ω as the paper moves along horizontally and the vertical height of each phasor traces out a sinusoid as a time-graph. Does that sound a nicer way to think of it? Presumably not. But it might help intuitively in some cases, particularly with rotating machines. If you don't understand a word of the above description, see Wikipedia on [Phasors] for two helpful moving pictures of rotating lines tracing out sinusoids. Just to annoy me, they've made the

[2] We'll sometimes sloppily say 'frequency' when we really mean angular frequency: it's shorter.

sinusoid vertical and thus the horizontal displacement is what determines the value of the sinusoid at each time: the other way is more common in other sources that I've seen.

3.2 Mathematically

[Incomplete.]

A sinusoidal time-function can be represented by the real or imaginary part of $\exp(\mathrm{j}(\omega t + \alpha))$; remember that $\exp(\mathrm{j}\phi) = \cos(\phi) + \mathrm{j}\sin(\phi)$.

This has the convenience that changes of magnitude *and* phase can be made by simple multiplication by a complex number. For example, we can double the function and shift it to be 45° later in phase, by multiplying with the complex number $2\exp(-\mathrm{j}\pi/4)$ which could otherwise be called $2\underline{/-\pi/4}$.

Then, as all the quantities have the same ωt part, we can do a transformation by dividing out common factor of $\exp(\mathrm{j}\omega t)$.

More: Fourier-series, convolution versus frequency-domain multiplication ...

3.3 Steady-state solution of ODEs

[Incomplete.]

Consider writing ODEs for a systems of lots of capacitors and inductors, expressed in terms of some sought variable, and with a sinusoidal forcing function due to a source. The differential equation will be in terms of the sought variable and its time-derivatives: these will be equated with the forcing function. If you have to satisfy an equation like $y''' + k_2 y'' + k_1 y' + k_0 y = cos(\omega t)$, a good choice to start with for $y(t)$ could be a cosine or sine ... these have derivatives that are also cosines or sines, so they have a chance that parameters can be found that will satisfy the ODE (more conveniently, use exponentials).

This perhaps helps to show why it's reasonable to expect that the forced response of a linear system to a sinusoid is also a sinusoid.

3.4 Why complex numbers

If we accept the need of representing magnitude and phase of the sinusoidal quantities in a circuit, then why are complex numbers useful?

Think of KCL and KVL. Here we have to add the sinusoidal quantities. Consider the case of adding add $A\cos(\omega t + \alpha) + B\sin(\omega t + \beta)$. We could do this by splitting each function into a sum of pure cosine and sine, then adding these. Complex numbers represent this very neatly: the rectangular (kartesisk) form is directly suited to adding and subtracting. The same is true of vector addition. However, vector multiplications do not have the useful properties that complex number multiplications have, for impedance and power.

Think of the relation of voltage and current, for a capacitor or inductor, perhaps in combination with some resistors. A resistor's voltage and current phasors have the same angle, and the ratio of their magnitudes is described by the resistance. The capacitor and inductor likewise determine a ratio of voltage and current magnitudes, but they also cause a phase-shift (change) between current and voltage. A combination of resistors, capacitors and inductors can cause the voltage and current phasors to have phase-shifts anywhere between $\pm -90°$.

Impedances are not phasors. They do not represent sinusoidal time-functions. They are complex numbers that multiply or divide phasors to give other phasors. Like a resistance, an impedance changes the dimension and gives a scaling: a $5\,\Omega$ resistor converts e.g. 2 A to 10 V. Unlike a resistor, an impedance can also change the phase, by the convenient properties of complex numbers. A resistor and inductor together could make an impedance of $(4 + 3\mathrm{j})\,\Omega$ which can alternatively be written as $5\underline{/37°}\Omega$. It therefore converts a current of $2\underline{/-15°}$A to a voltage of $10\underline{/22°}$V.

The convenient properties for power will be seen in Topic 11. Complex numbers "could have been made for ac circuits".

4 Phasors

4.1 The 'phase-vector'

It is very common to draw phasors as lines like vectors. One way would be like an Argand diagram in the complex plane, with every phasor starting at the origin (0,0). A more common choice is to show additions like vectors, by joining phasors end to end. For example, KCL and KVL can be expressed by expecting that all the phasors in the law should be able to be put head to tail to form a closed loop (zero sum), if all defined in the same direction in the circuit.

These *phasor diagrams* can start looking quite complicated. Sometimes the equations can seem simpler to look at! But sometimes the diagram is very helpful at giving a feeling for the sensitivity of a circuit to changes in a parameter.

4.2 Phase reference

An ac analysis with phasors inherently assumes a particular frequency, e.g. ω, for all quantities. Time-functions have only two other degrees of freedom: the amplitude and phase angle. The choice of phase reference means the choice of which time-function would be seen as a phasor of zero phase (purely real).

It is an arbitrary choice. You could choose that your reference is $\sin(\omega t)$ as we did earlier. Then a time-function $I\sin(\omega t + 5°)$ could become a phasor $I\underline{/5°}$. (I say 'could' instead of 'would' because of the further arbitrary choice of how to scale the magnitude: see below!)

Or you could choose that the reference is $\cos(\omega t)$, which is quite conventional, particularly in signals and communication subjects. (It is liked because $\cos(\phi) = \Re\{\exp(\mathrm{j}\phi)\}$.) In that case, the time-function $I\sin(\omega t + 5^\circ)$ would become $I\angle{\frac{-\pi}{2} + 5^\circ} = I\angle{-85^\circ}$, due to the 90° phase difference between sine and cosine.

These are not the only possible references: one could choose an arbitrary angle such as $\cos(\omega t + -134^\circ)$. A good choice can help to simplify the arithmetic, just like a good choice of earth node in a nodal analysis. The important thing is that the same reference must be used for all independent sources, and must be used if converting solutions back to time-functions.

4.3 Scaling: magnitude

We have used the terms amplitude and magnitude. *Amplitude* suggests 'bigness', but more specifically it's typical when describing the height of a wave, such as a sinusoid: it seems well suited to talking about sinusoidal time-functions. *Magnitude* is common when talking of complex numbers. We will use this to describe the absolute value of a phasor or impedance.

The magnitude of a phasor does not have to be equal to the amplitude of the time-function that it represents. We will see, in the Topic on ac power, that it is often convenient to use phasor magnitudes that are $\frac{1}{\sqrt{2}}$ of the amplitude of the time-functions! It simplifies power calculation. That is what we're used to in ac: your electrical goods at home may say they use electricity at 230 V, but this means the amplitude (peak value) is about 325 V. A further definition, sometimes used by people in communications subjects, is *peak-to-peak* amplitude, meaning the distance from negative to positive peaks, instead of from a peak to zero.

As with phase, the important thing about amplitude scaling is to be consistent. If you choose to multiply amplitudes of time-functions by some arbitrary factor like 3π to give the phasor magnitudes, then you should do this to all amplitudes that are used in your calculation. Your phasors must then be interpreted in that way: if you convert back to time-functions you must *divide* the phasor magnitudes by the same factor, in order to get results in the same units as the other time-functions. The ability to use arbitrary scaling factors is yet another thing based on the assumption of linearity of the circuit.

5 Impedance and Admittance

Resistors, inductors and capacitors are all seen in a very similar way in ac circuits. Each of them can be represented as a single complex number called an *impedance* (sv: *impedans*), denoted Z. The impedance is used in just the same way as resistance in dc circuits: it describes the ratio of voltage and current phasors for the component. The reason it is complex is that there may be a phase-difference between the voltage and current: this is then shown by the angle of the complex number for impedance.

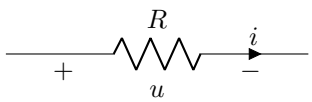
In the following subsections it will be shown that a resistor has a purely real impedance (still called resistance), and a capacitor or inductor has a purely imaginary impedance (which can be called *reactance* (sv: *reaktans*), denoted X).

Just like resistors in dc circuits, the impedances can be combined into a single "equivalent" when components are connect in series or parallel. The equivalent impedance of multiple components may have any combination of real and imaginary parts, although the real part will be positive if the resistors are true resistors (that do not have negative resistance).

It is sometimes convenient to use the reciprocal of impedance, called *admittance* (sv: *admittans*) and denoted Y. Similarly to conductance in dc circuits, admittance is convenient for addition of components in parallel connection, as it describes how easily current can flow.

5.1 Resistance stays resistance

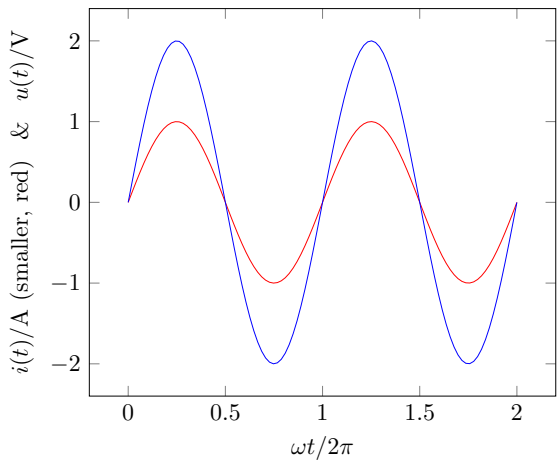
It's easy to start with a resistor. Nothing has changed since the dc circuits: at any point in time, there is proportionality of voltage and current. If a sinusoidal current $i(t) = \hat{I}\sin(\omega t)$ is put through a resistor R,



then the voltage is

$$u(t) = Ri(t) = R\hat{I}\sin(\omega t).$$

The two sinusoids are clearly "in phase" with each other: they have the same angle. The following plot is an example where $R = 2\,\Omega$.



If this sinusoidal current $i(t)$ is represented in its phasor form as $i(\omega) = \hat{I}\angle{0}$, then we see that the voltage

$u(t)$ would be represented as

$$u(\omega) = R\hat{I}\underline{/0}.$$

The ratio of these phasors is the impedance of the resistor,

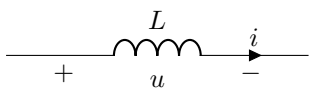
$$Z_{\text{resistor}} = \frac{u(\omega)}{i(\omega)} = R,$$

... which is just its resistance. It's nice and simple for a resistor.

Notice that even if we had used a different reference for making the phasor of current, the resulting impedance would be the same. For example, if a cosine reference had been used, then the current phasor would have been $i(\omega) = \hat{I}\underline{/-\pi/2}$, *and* the voltage phasor would have been $u(\omega) = R\hat{I}\underline{/-\pi/2}$, so the ratio would still be a real number.

5.2 Inductance becomes reactance

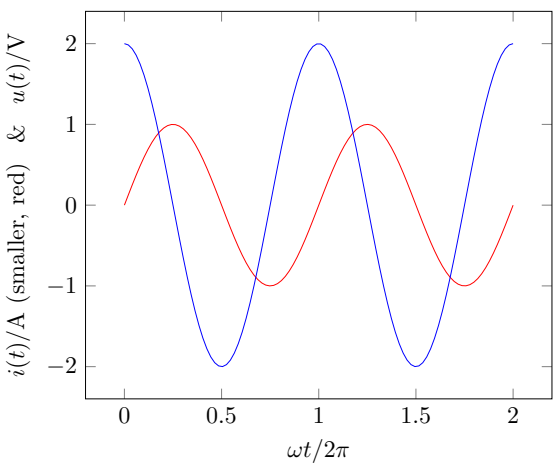
If now the sinusoidal current $i(t) = \hat{I}\sin(\omega t)$ is put through an inductor L,



then the voltage

$$u(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t} = \omega L\hat{I}\cos(\omega t).$$

The two sinusoids are now a sine and cosine: there is exactly a 90° phase-shift between them. The voltage is 'leading' the current, which means that the 90° is how much earlier in time the voltage reaches (for example) its positive peak compared to the current. The following plot is an example, where $\omega L = 2\,\Omega$.



With the sinusoidal current $i(t)$ is represented as the phasor $i(\omega) = \hat{I}\underline{/0}$, the voltage $u(t)$ would be represented as

$$u(\omega) = \omega L\hat{I}\underline{/\tfrac{\pi}{2}}.$$

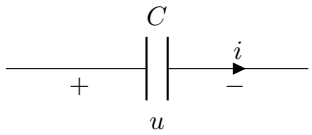
The ratio of these phasors is the impedance of the inductor,

$$Z_{\text{inductor}} = \frac{u(\omega)}{i(\omega)} = \mathrm{j}\omega L = \mathrm{j}X_{\mathrm{L}}.$$

This is an impedance that is purely positive imaginary, and is proportional to frequency. That means that at higher frequency it becomes harder (requires more voltage) to force a given amplitude of sinusoidal current through an inductor. Think of the inductor as an inertia (tröghet) to current: at higher frequency the current has to change more quickly, so more push (voltage) is needed.

5.3 Capacitance becomes reactance

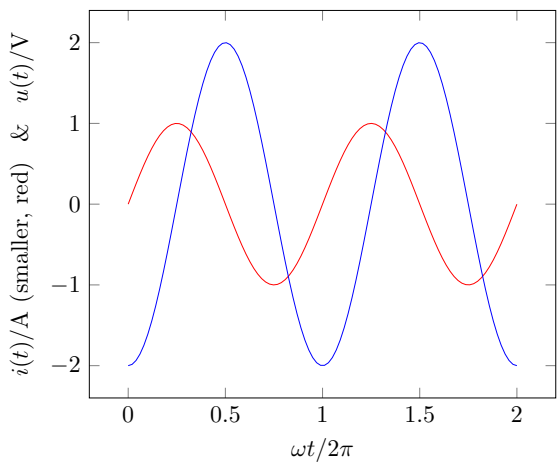
Finally, let's put the sinusoidal current $i(t) = \hat{I}\sin(\omega t)$ through a capacitor C,



in which case the voltage is

$$u(t) = \frac{1}{C}\int i(t)\mathrm{d}t = \frac{-1}{\omega C}\hat{I}\cos(\omega t).$$

The two sinusoids are again a sine and cosine, but the cosine is negated (from integrating the sine). There is still an exact 90° phase-shift, but this time the voltage is 'lagging' the current: it reaches (for example) its positive peak 90° *later* than the current. The following plot is an example, where $\frac{1}{\omega C} = 2\,\Omega$.



With the sinusoidal current $i(t)$ is represented as the phasor $i(\omega) = \hat{I}\underline{/0}$, the voltage $u(t)$ would be represented as

$$u(\omega) = \frac{1}{\omega C}\hat{I}\underline{/\tfrac{-\pi}{2}}.$$

The ratio of these phasors is the impedance of the capacitor,

$$Z_{\text{capacitor}} = \frac{u(\omega)}{i(\omega)} = \frac{1}{\mathrm{j}\omega C} = \frac{-\mathrm{j}}{\omega C} = \frac{-\mathrm{j}}{X_{\mathrm{C}}}.$$

This is an impedance that is purely *negative* imaginary, and is *inversely* proportional to frequency. That means that at higher frequency it becomes easier to force a given amplitude of sinusoidal current through a capacitor. Think of the capacitor as a bucket (of charge) that moves back and forward on each cycle of the sinusoid: the rate of transfer (charge/time) is greater if the movements happen in the shorter time (higher frequency).

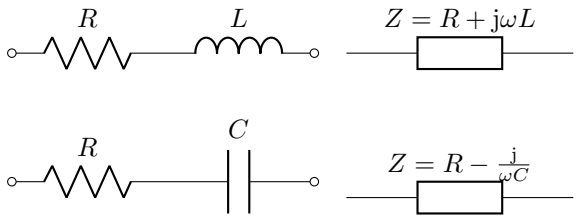
5.4 Reactance, Susceptance, and more

First, a little about reactance, which we have seen in the preceding subsections. What seems conventional in power-related subjects[3] is that reactance is a *positive* and *real* number: inductive reactance is $X = \omega L$, and capacitive reactance is $X = 1/(\omega C)$. Then we have to put the imaginary unit and correct sign next to the reactance when we calculate by adding resistances and reactances together.

Sometimes it is convenient to work with conductance, $G = 1/R$ instead of resistance, such as when adding parallel resistors or looking at current-to-voltage ratios. Likewise, it may sometimes be convenient to work with susceptance, $B = 1/X$, instead of reactance.

5.5 Combining impedances

When a resistance and reactance are connected in series, the ratio of voltage/current across the series combination is some complex number called the *impedance*. A component called an impedance can be used for the general case, to represent a resistor, capacitor, inductor or some two-terminal combination of two or more of these components. Below is the impedance that represents a resistor and inductor in series, at angular frequency ω.



A proof of the equivalent resistor for series and parallel connection of resistors was provided in Topic 02. The exact same argument can be used for impedances, using ac analysis. Impedances in series add: $Z = R + \mathrm{j}X$ for an inductor and resistor, or $Z = R - \mathrm{j}X$ for a capacitor and resistor. Admittances in parallel add: $Y = G - \mathrm{j}B$ for in inductor and resistor, or $Y = G + \mathrm{j}B$ for a capacitor and resistor. Combinations of an inductor and capacitor have a purely imaginary impedance, of a sign that depends on which one "wins", i.e. which series impedance is bigger, or which parallel impedance is smaller. This is considered further in the Topics concerning filters and power-factor correction.

Any group of resistors, inductors and capacitors that has just two terminals connecting to it can be represented in an ac circuit as a single impedance, and needs at most two components to make an equivalent model. This can be found by calculating the impedance, then finding one or two components that provide the same impedance. It is good to become familiar with the signs: if you need a positive imaginary part in an impedance, then you need an inductor; a negative imaginary part needs a capacitor.

Conversion between a parallel and series combination of two components is often useful. It is a special (simple) case of the above conversion. Take, for example, a parallel resistor and capacitor, R_p and C_p. Their combined impedance Z_p is related to the component values by $\frac{1}{Z_\mathrm{p}} = \frac{1}{R_\mathrm{p}} + \mathrm{j}\omega C_\mathrm{p}$, so

$$Z_\mathrm{p} = \frac{1}{\frac{1}{R_\mathrm{p}} + \mathrm{j}\omega C_\mathrm{p}}$$

If we want to find a *series* pair of components that is equivalent to the parallel pair (at a particular frequency), then we see that a series resistance and reactance will sum to $Z_\mathrm{s} = R_\mathrm{s} \pm \mathrm{j}X_\mathrm{s}$. By comparing separately the real parts and imaginary parts, between the expressions for the series and parallel components, R_s and X_s can be found.

The above expression has the complex part in the denominator, which is not convenient for separating real and imaginary parts. Rewrite this after multiplying top and bottom by the complex conjugate, and equate it to Z_s:

$$\frac{\frac{1}{R_\mathrm{p}} - \mathrm{j}\omega C_\mathrm{p}}{\frac{1}{R_\mathrm{p}^2} + \omega^2 C_\mathrm{p}^2} = Z_\mathrm{p} = Z_\mathrm{s} = R_\mathrm{s} - \mathrm{j}X_\mathrm{s}.$$

The impedance has a negative value of reactance, so we need a capacitor in the series circuit: this is a general rule, that the parallel and series pairs need the same types of component, but just different values.

The real and imaginary terms can now be separated and equated. The real parts directly show the resistor,

$$R_\mathrm{s} = \frac{\frac{1}{R_\mathrm{p}}}{\frac{1}{R_\mathrm{p}^2} + \omega^2 C_\mathrm{p}^2} = \frac{R}{1 + \omega^2 C_\mathrm{p}^2 R_\mathrm{p}^2}.$$

The imaginary parts of the series and parallel combination must also be equal in order to achieve equivalence,

$$-\mathrm{j}X_\mathrm{s} = \frac{-\mathrm{j}}{\omega C_\mathrm{s}} = \frac{-\mathrm{j}\omega^2 C_\mathrm{p} R_\mathrm{p}^2}{1 + \omega^2 C_\mathrm{p}^2 R_\mathrm{p}^2},$$

[3] I have a distant memory that people in some subjects treat reactance as being a signed, imaginary quantity: they would write that the inductive reactance of an inductor L is $X = \mathrm{j}\omega L$, and of a capacitor is $X = -\mathrm{j}/(\omega C)$. But perhaps that's just a false memory: I can't find evidence of it now. Wikipedia agrees with the real and 'conventionally' positive definition. Often there are various, contradictory, conventions that one has to be careful about, between different branches of engineering/physics/etc. You can define anything you like, but it's helpful to follow the fashion within the subject you work in.

from which we see that the series reactance needs to be a capacitor,

$$C_\mathrm{s} = \frac{1 + \omega^2 C_\mathrm{p}^2 R_\mathrm{p}^2}{\omega^2 C_\mathrm{p} R_\mathrm{p}^2}.$$

The above is in fact a particular case of finding an equivalent impedance of a two-terminal circuit. This particular case is often useful. Conversion between series and parallel equivalent impedances will be relevant during the Topic on ac power.

6 Multiple sources and Superposition

6.1 A note on dependent sources

Recall, from dc superposition, that only the independent sources are treated as *driving* the circuit. The dependent sources only have any value if something is stimulating their controlling variable.[4] A dependent source is in this way a generalised resistor. Like a resistor, the component value is a ratio between two circuit quantities, and there is no particular voltage or current determined by the source itself without some stimulation from outside. Unlike a resistor, the two quantities are not limited to be a voltage and current pair, and they can be in different parts of the circuit.

In ac circuits, this fundamental point is of course still true: a dependent source only responds to the stimulus imposed by independent sources. A dependent source has no 'frequency' of its own. If the controlling variable is stimulated by an independent source at 50 Hz, the dependent source will have a 50 Hz output. If the controlling variable is stimulated by various independent sources of different frequencies, then it will respond to all of these. The dependent source's value (the quantity that multiplies the controlling variable) could be a frequency dependent complex number. This would mean that there is a a phase-shift as well as a scaling factor (gain) between the controlling variable and the source's output, and that this relation is a function of frequency. That is highly plausible behaviour for practical implementation of a dependent source! It is not difficult to include in the analysis: as usual, the same steps are followed as with dc analysis, but the numbers are complex. However, we normally don't bother going this far in our courses! As far as I'm aware, no exam has had a dependent source with a value that isn't just a real constant.

[4] A note about the claim that dependent sources only have a value if stimulated. One can make circuit diagrams where a dependent source would drive its own controlling variable. In our idealised circuits, the solution of zero (for all voltages and currents) should still be found when there are no independent sources. With dependent sources in combination with reactive components it would be possible to make circuits where an initial non-zero value causes an oscillation or unstable growth, that continues even in the absence of independent sources. But we will *assume* the circuits are stable and have no steady-state values except what the independent sources have caused. That is a basic assumption of ac analysis, where we assume the natural response has died away and only the forced response remains.

In the following subsections, "source" will be used to mean independent sources. When superposition is used, any dependent sources would be kept in the circuit for while solving each of the different superposition states caused by the different independent sources. This is just the same as in dc analysis.

6.2 Sources with the same frequency

When sources have the same frequency, they can be described as phasors within a single calculation. The same phase reference must be used for both, so that the phasors have the right relative phase.

For example, take two sources, of $U_1(t) = A\cos(\omega t + \alpha)$ and $U_2(t) = B\sin(\omega t + \beta)$. If we decide to convert to phasors with a cosine reference and with peak values, then certainly $U_1(\omega) = A\angle\alpha$. But then we can't just decide to use another reference, such as sine, for U_2; the function $U_2(t)$ has to be converted to a phasor also using peak values and cosine reference, $U_2(\omega) = B\angle\beta - \frac{\pi}{2}$. Check that! The $-\pi/2$ includes the shift needed to make a cosine look like a sine.

Having more than one source generally makes the arithmetic significantly more awkward. That's no issue if working with computers and numbers; and it's still often easier than using superposition.

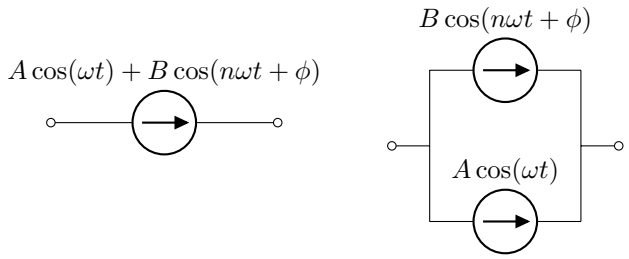
6.3 Sources with different frequencies

When sources have different frequencies, they cannot be represented by phasors that have a meaning when used together.[5] Instead, a separate calculation is needed for each frequency. The time-functions of circuit quantities due to sources at each frequency can be calculated, and then these time-functions can be summed together for the complete time-function. This is an application of superposition. Superposition has to be used in this case of different frequencies, and results have to be converted back to time functions before being summed. (Superposition is still possible when all sources have the same frequency, but then it is a choice as an alternative to nodal analysis or other methods.)

6.4 A sources with multiple frequencies

A single source may be defined as having a time-function that is a sum of sinusoids at different frequencies. But this is equivalent to having one source for each of these frequencies. The following diagram shows equivalent ways of getting a two-frequency current. (Puzzle: what equivalent connection would be needed on the right if the sources were all voltage sources?)

[5] Remember that a phasor represents the size and phase of a sinusoidal function at a specific frequency. Sinusoidal functions at different frequencies do not have any steady phase-relation, as the relative positions of peaks or zeros between the functions are constantly changing with time.



Since a multi-frequency source can be represented as separate sources, it can be handled in the same way as above, using superposition for calculating the the quantities caused by separate sources with different frequencies.

6.5 DC is a frequency

It was mentioned much earlier that dc is a special case of ac, where the frequency is zero. In a fourier series the zero-frequency component is one of the parts.

We can get a dc solution in ac analysis by setting $\omega = 0$. It is fun to note that the method for finding the equilibrium "with constant sources" of a circuit with capacitors and/or inductors will come out automatically from the ac analysis when $\omega \to 0$. We replaced capacitors with open circuits: a capacitive impedance becomes infinite as $\omega \to 0$, which indicates an open circuit. Likewise inductors become short circuits, as their impedance drops to zero.

Therefore, if a circuit is stimulated by ac and dc sources, the steady-state solution can be found by superposition of the time-functions of voltage and current due to the different sources. This is just a particular case of the superposition with sources of different frequency.

6.6 Superposition for transients too

The assumption in ac analysis is that the circuit is in sinusoidal steady state. Mathematically (in differential-equation terminology), the transient response of the dynamic system has died away, and the only remaining activity is the forced response due to the sinusoidal forcing functions of the independent sources.

The *complete response* is the combination of the transient and forced response. It is what would be seen if one newly switched on a sinusoidal source to a circuit.

A convenient way to find the complete response is to find the forced response by ac analysis. Then find the state of the continuous variables at the point when the ac sources will start giving their output. At that starting time, the difference between the continuous variables from these two calculations is the amount that should be applied to calculate just the unforced transient response that should be added in time to the ac forced response in order to find the complete response.

6.7 Superposition, Linearity and Power

A warning or reminder: superposition assumes linearity. In a linear circuit we expect voltages and currents to vary in direct proportion to the sources that drive them. The preceding subsections used various forms of superposition with sources of different frequency, zero frequency, etc. This assumes that solutions are circuit quantities of current, voltage, potential or charge in a linear circuit. Power generally does not have a linear relation to these circuit quantities: for example, we know the power in a resistor is proportional to the square of the current (to avoid thinking of ac power, let's consider the power at some instant). We therefore should not generally try superposition with powers. There is a special exception to this, which will be in Topic 11.

7 Two-terminal equivalent sources

The Thevenin or Norton equivalent reduces a linear circuit that is "seen through two terminals" to be just a very simple circuit of a source and resistor, with the exact same behaviour.

This is not valid for transient calculations: the transient response of a circuit that includes L and C components, seen between two terminals, could be a complicated mixture of oscillations, and is dependent on the inital states. This cannot be simplified into two basic components.

We *did* use equivalents within the transients calculations, but we carefully made the equivalent of only the part of the circuit that did *not* contain a capacitor or inductor! That was valid.

In ac analysis the equivalent circuits *are* valid. For *one particular frequency*, e.g. angular frequency ω, a two-terminal circuit has a particular combination of a source and impedance that is equivalent. The calculation method is exactly as for dc circuits, except that phasors and impedances are used instead of purely real-valued voltages and resistances.

The only extra complexity compared to dc is if we are asked to show the components that make up the Thevenin or Norton impedance. If Z has positive non-zero real and imaginary parts then a resistor and inductor are needed in order to make this impedance. If Z has positive real and negative imaginary parts then a resistor and capacitor are needed. Purely real or purely imaginary Z can be made from a single component. A negative real part of Z suggests negative resistance, which should not be possible unless there are dependent sources in the circuit. In such cases we can show a negative resistance component in the equivalent circuit model.

8 Complex Numbers

8.1 Notation

Consider a complex number z.[6]

In *rectangular form* (sv: *kartesisk*), let us say that

$$z = a + \mathrm{j}b,$$

where a is the real part, b is the imaginary part, and $\mathrm{j} = \sqrt{-1}$.

The real and imaginary parts of a complex number z, or of some larger expression, can be written as

$$\Re\{z\} \quad \text{and} \quad \Im\{z\}.$$

In *polar form* (sv: *polärform*),

$$z = m\mathrm{e}^{\mathrm{j}\phi},$$

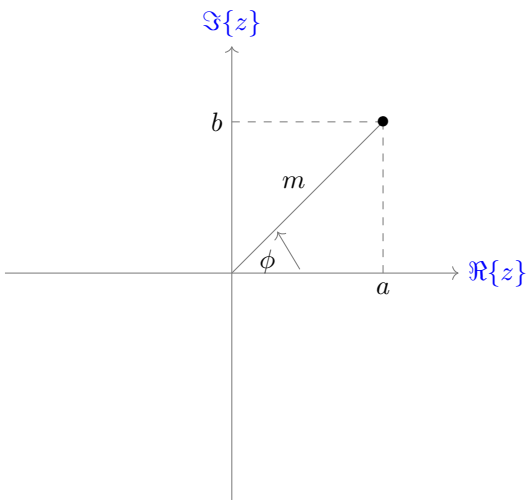
where m is the magnitude (or *absolute value*) and ϕ the angle (or *argument*).

The magnitude and angle can be written as

$$|z| \quad \text{and} \quad \angle z.$$

In the canonical case, a and b are real, m is real and positive, and ϕ is real and could be defined either as positive $0 \leqslant \phi \leqslant 2\pi$ or else within the range $-\pi \leqslant \phi \leqslant +\pi$.[7]

The above definitions are conveniently plotted "in the complex plane" as an *Argand diagram*, as shown below. The position of the complex number is here shown by a bullet.



[6] Here, the lower-case z is used to denote a complex number in general, not a impedance (which usually is denoted by an upper-case Z). This is conventional, similar to the use of x for an unknown real number.

[7] Of course, we could still define a complex number z by using complex values of a or b, or negative or complex m, or values of ϕ redefined as $\phi + 208\pi$, etc! The restrictions are chosen to show a "canonical form" about which we can more easily develop rules, because of assuming these restrictions. For example, if we see $a - \mathrm{j}b$, and know that a and b are positive reals, we can assume the number lies in the 4th quadrant of the complex plane.

The four parts between the axes are the quadrants: the first quadrant is where both axes are real (top right), then the others are numbered anticlockwise. As the angle ϕ increases, the position of the complex number moves anticlockwise in the complex plane.

Negation of a complex number can be shown in various ways:

$$-z = -(a+\mathrm{j}b) = -a-\mathrm{j}b = -m\underline{/\phi} = m\underline{/\phi+\pi}.$$

Conjugation is shown as z^*. It means that just the imaginary part is negated, so that the number appears to be reflected in the real axis when seen in the complex plane,

$$z^* = (a+\mathrm{j}b)^* = a-\mathrm{j}b = m\underline{/-\phi}.$$

An exponential with an imaginary argument gives a complex number with magnitude of 1 and angle of the argument's magnitude:

$$m\underline{/\phi} = m\mathrm{e}^{\mathrm{j}\phi} = \mathrm{e}^{\ln(m)+\mathrm{j}\phi} = m\cos\phi + \mathrm{j}m\sin\phi.$$

8.2 Rectangular and Polar conversion

Looking at the Argand diagram shown in Section 8.1, we can think of converting between forms of the complex number. The real and imaginary parts are perpendicular, so trigonometry with right-angled triangles is our tool.

8.2.1 Polar to real and imaginary

The real and imaingary parts, a and b, are sides of a right-angle triangle with magnitude m as its hypotenuse; the side a is the one adjacent to the angle ϕ. Therefore, the rectangular parts can be obtained from the polar form as

$$a = m\cos\phi, \quad \text{and} \quad b = m\sin\phi.$$

This works for a number anywhere in the complex plane.

8.2.2 Rectangular to magnitude

Pythagoras' theorem can be used to find the magnitude:

$$m = \sqrt{a^2 + b^2}.$$

This also works for a number anywhere in the complex plane.

8.2.3 Rectangular to angle

If the real part, a, of a complex number is positive, then the angle must be in the "right half-plane", i.e. $-\pi < \phi < \pi$. In that case, the angle can be found from

$$\phi = \tan^{-1}\left(\frac{b}{a}\right) \qquad (a \geq 0).$$

But be careful! The inverse tangent function should not be used like this for a complex number with a negative real part! Such a number is in the left half-plane. The angle calculated by the above equation will be wrong.

Consider a simple case of the number $1+1\mathrm{j}$, which has an angle of $\tan^{-1} 1 = 45°$. Now consider its negation, $-1-1\mathrm{j}$, which will be diametrically opposite it in the complex plane, at 225°, or equivalently at −135°. Bearing in mind that $\frac{-1}{-1} = \frac{1}{1}$, the result of taking the inverse tangent of the ratio of imaginary to real parts is again 45°.

From this we see that if the real part is negative, we should add or subtract 180° to the result from the tangent:

$$\phi = \tan^{-1}\left(\frac{b}{a}\right) + \pi \qquad (a < 0).$$

8.3 Polar-form arithmetic

Consider two complex numbers in polar form, $A\angle\alpha$ and $B\angle\beta$, where A and B are positive real numbers and α and β are real numbers.

There is a *very* convenient relation, de Moivre's theorem, for the product or quotient of polar complex numbers:

$$A\angle\alpha \cdot B\angle\beta = AB\angle\alpha+\beta,$$

and

$$\frac{A\angle\alpha}{B\angle\beta} = \frac{A}{B}\angle\alpha-\beta.$$

One usefulness of these is the case where a phasor is multiplied or divided by an impedance or admittance, e.g. $u = Zi$, where all three numbers can be complex. Then we see that the magnitude of the voltage across the impedance is the product of the current's magnitude and the impedance magnitude, and the phase of the voltage is the phase of the current plus the phase of the impedance: $|u| = |Z||i|$ and $\angle u = \angle i + \angle Z$.

Another usefulness comes when dealing with powers (in the mathematical sense: potenser). For example,

$$z^2 = |z|^2\angle 2\angle z,$$

or going in the other direction,

$$\sqrt{z} = \sqrt{z}\angle\angle z/2.$$

We will meet products such as uu^*, particularly when studying ac power. This simplifies nicely:

$$uu^* = |u|^2\angle\angle u - \angle u = |u|^2\angle 0 = |u|^2.$$

One more common case we will meet is a division by a complex conjugate, such as $\frac{|u|^2}{Z^*}$. The conjugate negates the angle of Z, but the division also negates the angle, so the angle of the resulting complex number is the same as the angle of Z,

$$\frac{|u|^2}{Z^*} = \frac{|u|^2}{|Z|}\angle\angle Z.$$

Addition and subtraction in polar form are not nice. In the general case, where we have some arbitrary angle, it is necessary to split the complex number into rectangular parts using cosine and sine functions. These rectangular parts can then be added or subtracted separately. In a few special cases we are lucky, when the angle is one where the cosine and sine have convenient values like 1, $1/\sqrt{2}$, $\sqrt{3}/2$, $1/2$, or (best of all) 0.

8.4 Rectangular-form arithmetic

Addition and subtraction in rectangular form are trivial: the real and imaginary parts are just handled separately as real numbers.

Multiplication of complex numbers in rectangular form is usually done by expanding out the terms,

$$(a+\mathrm{j}b)\cdot(c+\mathrm{j}d) = ac+\mathrm{j}ad+\mathrm{j}bc+\mathrm{j}^2bd = (ac-bd)+\mathrm{j}(ad+bc).$$

Division is a bit more awkward. The aim is usually to avoid complex quantities being on the bottom (denominator) of the expression. A complex number in the form $\frac{k}{a+\mathrm{j}b}$ cannot be immediately split into real and imaginary parts, whereas a number in the form $\frac{c+\mathrm{j}d}{k}$ can easily be split. We often want to find the real and imaginary parts of the number separately.

When a complex number is at the bottom of a quotient, we can always force this bottom part to become purely real by multiplying the top and bottom by the complex conjugate. Then it looks messy, but it can at least be easily split:

$$\frac{k}{a+\mathrm{j}b} = \frac{k}{a+\mathrm{j}b}\cdot\frac{a-\mathrm{j}b}{a-\mathrm{j}b} = \frac{k}{a^2+b^2}(a-\mathrm{j}b).$$

The bottom part was simplied by expanding and the relation $-\mathrm{j}^2 = 1$: $(a+\mathrm{j}b)(a-\mathrm{j}b) = a^2-\mathrm{j}^2b^2 = a^2+b^2$.

8.5 Choices of form during a solution

Rectangular form is very nice for adding, and a bit tedious for muliplying or dividing. Polar form is very nice for multiplying or dividing. But it's horrid for adding, in the general case where numbers $A\angle\alpha$ and $B\angle\beta$ may both need to be split into rectangular parts for adding, e.g. $A\cos(\alpha) + B\cos(\beta)$ and $\mathrm{j}A\sin(\alpha) + \mathrm{j}B\cos(\beta)$ ... consider that α or β may also be long expressions, and that the cos() parts will remain in the result unless you have lucky angles such as 60° or 90°, etc, that can be simplified to $1/2$ or 1.

The main advice is that you should not usually convert rectangular form to polar form. For some of the calculations — like when we want to get a time-function at the end — we will need to get the final

answer in polar form. But if we convert expressions to polar form too early, we might have to use sin() and cos() to make them rectangular again for addition or subtraction, which risks giving unpleasant expressions. So convert to polar form only when you see that no more addition/subtraction is needed until the end!

8.6 Computers

A computer lets us avoid all the complex algebra, at least when we have numeric values. If you had written a program for dc analysis, in a language such as Matlab that supports complex numbers, then you could just use the program right away for ac analysis. It would be necessary only to express the C and L components as impedances, and the independent sources (at one frequency) as phasors.

Some important functions are:
Magnitude: `abs()`
Phase: `angle()`
Phase in degrees: `angle()*180/pi`
Inverse tangent: `atan()`

The inverse tangent shouldn't normally be needed. The `angle()` function operates directly on a complex number, and sorts out the little details such as whether $\tan^{-1}\left(\frac{b}{a}\right)$ needs a phase-shift because of having a negative value of a.

If you have numbers, it's easy to do your ac calculations by computer, using short variables to represent the components. For example, `Z1`, `U` etc. It is good to work in small steps, defining new variables, so that you can check the intermediate steps for reasonableness. A few examples will be given in homeworks or exercises (not as required work, but as shown examples in the model solutions).

9 Summary of the AC introduction

This introduction to ac analysis has shown the use of complex numbers to do calculations for sinusoidal steady-state conditions on general circuits with voltage and current sources (independent or dependent) and resistors, inductors and capacitors.

As extensions of this method, we've considered superposition to extend to cases beyond pure ac. The solution with sources at different frequencies has to be done by adding the resulting time signals, since phasors at different frequencies can't meaningfully be directly added as complex numbers. Solutions from ac analysis can also be superposed with the natural response or with a dc solution, which also require superposition in time.

Although the calculations have involved phasors and impedances (which represent the circuit at a particular frequency), we have still seen a lot of *time* functions. In the later ac Topics there will not be much attention to time: calculations will be based on starting and finishing with phasors. When using these, we just accept that each complex number for voltage or current represents a sinusoidal quantity in time.

The typical type of exam question that comes up with relevance to this Topic is in the form "Here is a circuit with voltage source $u(t) = \cos(\omega t + \phi)$ and current source $i(t) = \sin(\omega t - \phi/2)$; solve for the marked voltage $u_x(t)$". You will need to decide whether the circuit can be represented as a single calculation by the phasor method, or whether the circuit does *not* have only one frequency in steady state, and therefore has to have separate analyses combined by superposition. Then you will need to convert sources and R, L, C components to the appropritate phasors and impedances, and make the phasor solution. Finally, you need to translate back again into "the time-domain" to find the requested sinusoidal time-function. This requires the magnitude and phase of the phasor that represents the requested quantity, and you have to use the same reference phase (e.g. choosing that $\cos(\omega t)$ corresponds to a phasor with zero phase). Don't forget to include the ωt term too, in the final cosine or sine function.

10 — Extra —

Quite a lot of the earlier parts of this Topic's notes was not strictly needed for doing calculations on ac circuits of the type that we see in the homework and exercises. These parts could have been put in this "Extra" section: the reason they weren't is that they might help with understanding the calculation part and its limitations. There's not much more to put here.

10.1 Pure phasors, ignoring time

We've been told that the rest of the course on ac will mainly ignore time. This is how people who need to analyse ac circuits actually work, in most cases. They don't care about some absolute time-point $t = 0$ and whether the signal is at a peak or a zero at that time (cosine or sine), nor about the "absolute" phase relative to some standard. They just care about relative phases within a single circuit. So all voltages and currents are treated as phasors from the start.

It is necessary to define what angle the phases are defined relative to. Usually we choose to define one of the voltages or currents as the zero phase; quite often we choose a voltage source. Then at least *one* of the phasors is a nice simple real number. The source values appear in many expressions, since the sources cause all the other currents and voltages, which is why it is often desired to define a source as the reference. The usual choice is whatever is most convenient for the calculation — just like the choice of a ground node.

There is, as usual, an exception to the above claim that only relative phase is important. In power systems it is increasingly common to have synchronised *measurement* of phasors relative to a common "zero time": in this case, the reference is a cosine (at the

systems's declared frequency, e.g. exactly 50 Hz or 60 Hz) with its zero phase being at the start of each second (in UTC time). The timing is often taken from a GPS receiver.

The `PMUs[link]` (phasor measurement units) that do this are a current "hot topic" in power systems, to make grid operators more aware of what is happening. See also a more commercial `PMUs[link]` page.

The phasor measurement is arguably not really an exception. I don't think many users care *directly* about how their voltage or current relates to UTC seconds! The time reference is just a way to permit the grid operator to collect results from measurements over grids that cover distances of a thousand (and more) kilometres, and still know the *relative* phases of measured voltages and currents in different places within fractions of a millisecond. So it's still relative phase that matters, but the time-reference is a convenient way to measure relative phase even across continental distances.