Electric Circuit Analysis, KTH EI1120
N. Taylor,

Topic 10: Frequency-response & filters

Starting from Topic 09, we have been finding and interpreting equations that are functions of frequency. Impedances and admittances contain $j\omega L$ or $j\omega C$ parts. Voltages and currents are sometimes written as e.g. $u(\omega)$, at least to remind us that we intend the variables as phasors rather than as time-functions or dc values.

We have looked at changing a component or input, like a capacitance or voltage, but not at the *effect* of frequency on how the circuit behaves. The frequency has mainly been assumed to be a given *constant* for the circuit we are studying. This is often reasonable, such as when calculating or designing powers and power-factor correction in an ac power circuit known to be running at 50 Hz or 60 Hz. In some other cases, we care about a circuit's behaviour over a range of frequencies, and might even want to design a circuit so that some voltage or current, or relation between voltages and currents, has a particular dependence on frequency.

The term *frequency response* describes how relations within an ac circuit can depend on frequency. Even for a simple case such as the relation of voltage to current on an inductor, we see from $u(\omega) = j\omega L i(\omega)$ that the ratio of the voltage magnitude to current magnitude is proportional to frequency, and the voltage is always leading the current by $\pi/2$. Circuits with several impedances can produce much more complicated frequency-response, and non-monotonic changes in amplitude and phase.

Sometimes the frequency response of an existing system or design is to be found; at other times, a particular frequency response is desired, and a circuit must be designed to achieve this.

In classic radio circuits, the tuner (the knob that you turn to choose a radio station) is a variable capacitor, which changes the resonance frequency of a type of filter circuit, so that it rejects frequencies other than the one narrow range that you want. Another non-power example is the circuits that split audio-frequency signals in a loudspeaker unit that contains a smaller loudspeaker (tweeter) optimised for higher audio frequencies, and a larger one (woofer) for lower frequencies. There are many other examples in general signals and communications subjects: a generic one is a "low-pass filter" to remove high frequency noise that can come into circuits from electric sparks, radio signals etc.

In power applications, non-sinusoidal currents are drawn by many modern electronic devices that use power-electronic components such as transistors and diodes. These currents can, by fourier analysis, be seen to contain many frequencies. A common requirement is to limit the size of the currents that are not at the intended frequency of the supply, such as 50 Hz; this can be done by filter circuits that either make it easy for the currents at unwanted frequencies to pass back to their source, or difficult for them to pass into the rest of the network.

In this Topic we consider network functions, amplitude response and phase response, and asymptotic plots of these responses (Bode diagrams). The strong relevance to the laboratory task is emphasised by analysing exactly that circuit!

1 Network function

For our purposes here, a *network function* (sv: *nätverksfunktion*) describes how one circuit variable (current or voltage) that we control affects another circuit variable that we measure.[1] The following circuit is a simple example. Its principle is behind many important practical situations, such as filtering of high-frequency noise, or undesired limitation of speeds in electronic equipment.
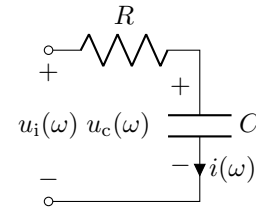
If the circuit is driven by an input voltage $u_i(\omega)$ applied to the terminals at the left, then the current $i(\omega)$ is

$$i(\omega) = \frac{u_i(\omega)}{R + \frac{1}{j\omega C}}$$

The network function relating this current to the voltage $u_i(\omega)$ that drives the circuit is defined as the ratio of their phasors,

$$H(\omega) = \frac{i(\omega)}{u_i(\omega)},$$

which, from the preceding equation, is seen to be

$$H(\omega) = \frac{i(\omega)}{u_i(\omega)} = \frac{1}{R + \frac{1}{j\omega C}} = \frac{j\omega C}{1 + j\omega CR}$$

This has dimension of admittance $[\Omega^{-1}]$, as is expected when the network function relates a current to a voltage.

We will see later that it can be helpful to rearrange these functions in a special form, like the final one shown above.

Good advice when handling network functions is to look for a simple way to solve the circuit, and *not* to

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[1] One definition of network function is that it's a purely steady state ac concept, relating phasors expressed in the variable $j\omega$; this distinguishes it from a *transfer function*, which relates Laplace transforms that are expressed in the variable $s = \sigma + j\omega$.

do any simplifications like expanding parentheses or forcing all complex parts to the top of the expression. These changes can make it harder to manipulate and interpret the network function.

The same circuit, above, could have other network functions defined. For example, the voltage across the capacitor is related to the input voltage by a different network function, which can be found easily by voltage division,

$$H(\omega) = \frac{u_\mathrm{c}(\omega)}{u_\mathrm{i}(\omega)} = \frac{\frac{1}{\mathrm{j}\omega c}}{R + \frac{1}{\mathrm{j}\omega C}} = \frac{1}{1 + \mathrm{j}\omega CR}.$$

This is dimensionless, as it ought to be for a ratio of voltages.

Replacing the capacitor with an inductor, something rather "opposite" could intuitively be expected. After all, at high frequencies a capacitor is a low impedance, and an inductor is a high impedance.
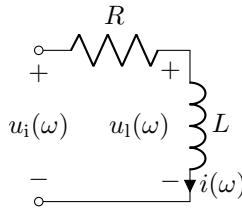
The ratio of voltages now gives a different network function,

$$H(\omega) = \frac{u_\mathrm{l}(\omega)}{u_\mathrm{i}(\omega)} = \frac{\mathrm{j}\omega L}{R + \mathrm{j}\omega L} = \frac{\mathrm{j}\omega L/R}{1 + \mathrm{j}\omega L/R}.$$

This looks more complicated than the function with a resistor and capacitor, but actually it produces plots with the same basic features, just shifted or reflected: compare the "low pass" and "high pass" graphs in the next section. We could avoid the ω appearing twice by rewriting the function as

$$\frac{\mathrm{j}\omega L/R}{1 + \mathrm{j}\omega L/R} = \frac{1}{1 - \mathrm{j}(R/L)/\omega},$$

but we generally prefer to stick to a few simple "building blocks", in which case we prefer to keep the denominators in the same form for both circuits.

"Duality" of course applies, so a similar set of functions could be found when swapping conductance with resistance, series with parallel, inductor with capacitor, and voltage with current.

2 Filters

To *filter* is to separate into different parts. An electric filter circuit is designed to have a network function that is frequency-dependent in a particular way. The result can be that it allows different frequencies in the input to come out of the filter with different gains (amplitude changes) or phase-shifts. One extreme case is when one wants to prevent all frequencies

in a particular range from getting into the output. Some practical applications of filter circuits have been mentioned in the introduction.

There are several common types of filter, with names based on parts of the spectrum (frequency range) that they "pass" or "stop". Four common ones are shown in the following figure. The word "band" refers to a range of frequencies not extending down to zero or up to very high values; "low" and "high" refer to the extremes of the frequency range.
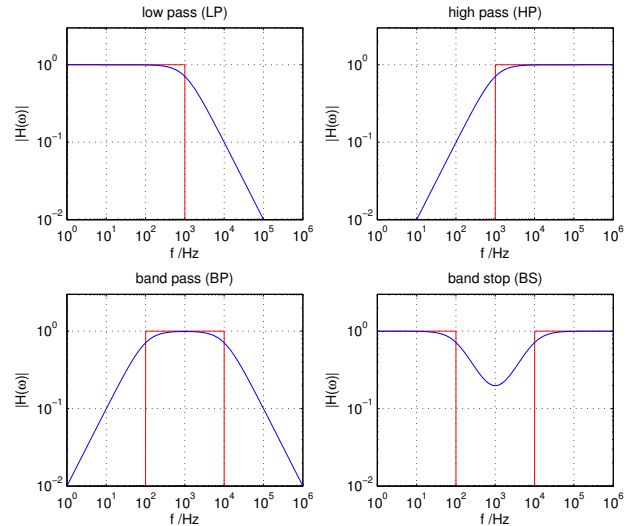
In each case the red lines show an "ideal" response, which cannot be attained in reality. The smooth blue curve shows a very simple implementation with a network function that could be made from one or a few simple circuit-elements.

A *low-pass* filter, for example (top left) has a flat response at low frequencies, so it allows these frequencies all to pass with a similar size. Above some frequency (which is $1\,\mathrm{kHz}$ in the example above) the network function reduces the relative strength of the signals.

3 Frequency response

A network function by itself is a complex function of frequency. For the sort of circuit we're looking at in this Topic, where we don't have resonances between inductors and capacitors[2], the network function can be written in a special form consisting of just a few basic terms. An example is this,

$$H(\omega) = k\frac{(\mathrm{j}\omega/\omega_0)\,(1 + \mathrm{j}\omega/\omega_3)}{(1 + \mathrm{j}\omega/\omega_2)\,(1 + \mathrm{j}\omega/\omega_1)}. \qquad (1)$$

The main terms in the numerator and denominator are each in the form $(1+\mathrm{j}\omega/\omega_x)$, where ω is the variable of angular frequency, and ω_x is a constant. These types of terms cannot be cancelled unless the ω_x happens to

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[2]Or similar forms of resonance caused by combining reactive and active components such as multiple capacitors and an opamp.

have the same value for a term on the numerator and the denominator.

The only other terms that might be needed are an overall gain that we can call k, and a term $j\omega/\omega_x$ (possibly squared or cubed etc) multiplying or dividing the whole function. Notice that there would not need to be terms like this in the numerator *and* denominator at the same time: they would then be able to be simplified into a constant that could be included in k. Notice also that the numerical values of ω_0 and k could be redistributed: in the above example, one could multiply both k and ω_0 by some constant without changing the function. In this way, k could be combined with ω_0 to make the equation look simpler. In actual network functions a term $j\omega/\omega_x$ is often found together[3] with a term $1 + j\omega/\omega_x$; in that case it is probably neater to keep both parts instead of combining k with one part. These are mainly matters of taste that you can decide about after some experience!

Each term in the numerator containing ω is called a zero of the function, and each term in the denominator containing ω is called a pole. This is from general terminology for complex functions; we don't need to consider why (although you could start looking [here!]) but just to recognise the names as alternatives for "things in the top" and "things at the bottom"!

For practical purposes, the most interesting features of $H(\omega)$, when a signal travels through different cables, amplifiers, filters etc., are usually the amplitude and then possibly the phase. The amplitude is particularly meaningful: it relates to the power available from the signal, and it can be measured by a voltmeter or ammeter at any point in a circuit. Phase is a relative measurement, for which we would have to compare to another measurement (such as the input to a circuit) or to a specific time-source (such as UTC seconds). It is therefore less common to be able to measure phase, or to care about it, although in some cases it is very important. The rectangular form, of real and imaginary parts, suffers the same problem as phase, of needing a reference for "what is the zero angle"; it has further problems of breaking up the interesting magnitude into different parts. Rectangular form is not a common way to think about any $H(\omega)$ function.

4   Plotting

Nowadays it is easy – and sensible – to plot a network function by writing it as an equation on a computer, then plotting the amplitude and phase of the solution at many frequencies.

But as usual, there are still some advantages to being able to visualise (in the imagination) the function, just from seeing the equation. This also helps when one has to work backwards, finding a suitable equation to describe a desired type of frequency response. Earlier generations of engineers *had* to use approximate methods, but we can try to take advantage of quick pen-and-paper methods *and* more detailed computer solutions. Plotting is useful for giving an immediate insight into the behaviour of the network function. Many products are specified by plots of their frequency-response in amplitude and/or phase, or by terminology that comes from such plots, e.g. "corner frequency".

Given a function such as (1), we can consider each of the separate multiplied terms, and how it behaves alone. Later we will see that combining their separate effects is surprisingly easy.

It is normal to plot in log-log scale for network function amplitude plots. One reason is simply that one typically studies amplitude responses and frequencies that vary over several "decades" (decade is used to mean "power of ten", sv: tiopotens). Another reason is that network functions typically contain powers[4] of frequency. A power function, such as $y = x^n$ becomes a straight line with gradient n when plotted in $\log y/\log x$ scale; We will see this soon — in fact, the graphs in Section 2 show that even the actual filter functions were mainly straight lines, but with curves joining them. Yet another reason for logarithmic scale is that we can then add the amplitudes found from the different terms in the network function, because $\log ab = \log a + \log b$.

Consider a term $j\omega/\omega_0$. Its amplitude is ω/ω_0. Its phase is always $+\pi/2$ ($+90°$), as $+j$ multiplies a real value. What does this term's magnitude look like in log-log scale? The magnitude is $|H(\omega)| = \omega/\omega_0$, which can be written as $\frac{1}{\omega_0}\omega^1$. It therefore appears as a straight line with gradient of 1 decade (y) per decade (x). At the point where $\omega = \omega_0$, the line passes through $|H(\omega)| = 1$, which in log-scale has the value $\log|H(\omega)| = \log 1 = 0$.

For the same type of term in the denominator, we have

$$\frac{1}{\frac{j\omega}{\omega_0}} = -j\omega_0\omega^{-1}$$

which has a constant phase of $-\pi/2$, and would have a gradient of -1, also passing through $|H(\omega)| = 1$ when $\omega = \omega_0$.

Now the other type of term is considered, where there is an additional real part: $(1 + j\omega/\omega_0)$.
When $\omega \gg \omega_0$, the 1 is negligible, so this term behaves like $j\omega/\omega_0$ which we studied before.
When $\omega \ll \omega_0$, this term becomes a constant of 1.
In between the two extremes, where $\omega = \omega_0$, the term is $(1 + j)$, which is $\sqrt{2}\big/\frac{\pi}{4}$.

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[3]Terms that arise together: as an example, consider the relation between the inductor voltage and total voltage in a series connection of resistor R and inductor L. By voltage division, this ratio is $\frac{j\omega L}{R + j\omega L}$, which can be written in the standard form as $\frac{j\omega L/R}{1 + j\omega L/R}$, where L/R could be written as $1/\omega_x$.

[4]Here, "powers" is in the sense "potens", not "effekt"!

When the term is in the denominator, so that we have $1/(1 + \mathrm{j}\omega/\omega_0)$, then:

When $\omega \gg \omega_0$, it is like $-\mathrm{j}\omega_0\omega^{-1}$ which we studied before.

When $\omega \ll \omega_0$, it becomes a constant of 1.

When $\omega = \omega_0$, it is $1/(1 + \mathrm{j})$, which is $\frac{1}{\sqrt{2}}\angle\frac{-\pi}{4}$.

5   Decibel scale

We have seen that a logarithmic scale on both axes is useful for amplitude plots, since it makes typical simple network functions be straight lines.

The *decibel* scale is a widely used logarithmic scale for comparing powers. The result is distinguished by the short name dB. For some ratio of power, P/P_0 the decibel value is

$$\mathrm{dB} = 10\log_{10}\left(\frac{P}{P_0}\right) \tag{2}$$

On the assumption that power is quadratically related to voltage (resistor), i.e. that $(P/P_0) = (U/U_0)^2$,

$$\mathrm{dB} = 10\log_{10}\left(\frac{U^2}{U_0^2}\right), \tag{3}$$

or equivalently

$$\mathrm{dB} = 20\log_{10}\left(\frac{U}{U_0}\right), \tag{4}$$

which is the usual formula for converting voltage or current to a decibel scale.

The "reference" U_0 or P_0 could be a chosen value. For example, the choice $P_0 = 1\,\mathrm{mW}$ (milliwatt) is used for a lot of specifications of communication equipment: in that case, the calculated powers in decibels are then expressed as dBm (decibels relative to a milliwatt), and voltages are often chosen to be relative to the voltage that is needed to give $1\,\mathrm{mW}$ in a $600\,\Omega$ resistor.

However, when we are expressing a *network function*, instead of a single value, in decibel scale, there is no need to consider a reference. The network function is already a relative measure: it is a ratio of circuit quantities.[5] So the decibel value of a network function can be found just by using the network function's magnitude as the "ratio" that we're converting to the dB scale,

$$\mathrm{dB} = 20\log_{10}|H|. \tag{5}$$

We must be careful to take the absolute value when putting it in computer programs: see the later example.

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[5]It must be admitted here that network functions relating the dissimilar quantities of voltage and current are not dimensionless. Then the decibel result is dependent on the units. A dimensional purist might like to divide the function by a unit, to make the argument to log() dimensionless. For our purposes, we'll just accept that all our network functions can be used as arguments to $20\log|H(\omega)|$.

6   Bode plots

A Bode amplitude plot is an "asymptotic approximation" of $|H(\omega)|$, using decibels plotted against the logarithm of frequency. A Bode phase plot shows phase against logarithm of frequency.

The functions we consider are split into factors that multiply in the top and bottom of the equations. Phase adds, logs add.

Sketching a Bode amplitude plot is a particularly common task. The best way is probably to learn what the plot of each commonly found term such as $(1 + \mathrm{j}\omega/\omega_x)$ or $1/(1 + \mathrm{j}\omega/\omega_x)$ looks like, then to draw these separately for each term, and then add these plots together.

For the amplitudes, in decibels, the four basic terms, $\mathrm{j}\omega/\omega_0$, $\frac{1}{\mathrm{j}\omega/\omega_0}$, $\frac{1}{1+\mathrm{j}\omega/\omega_0}$ and $(1 + \mathrm{j}\omega/\omega_0)$, are shown clockwise from top left in the following figure, where the frequency $1\times10^3\,\mathrm{Hz}$ (corresponding to the chosen ω_0 in this example) is marked with '$*$'.
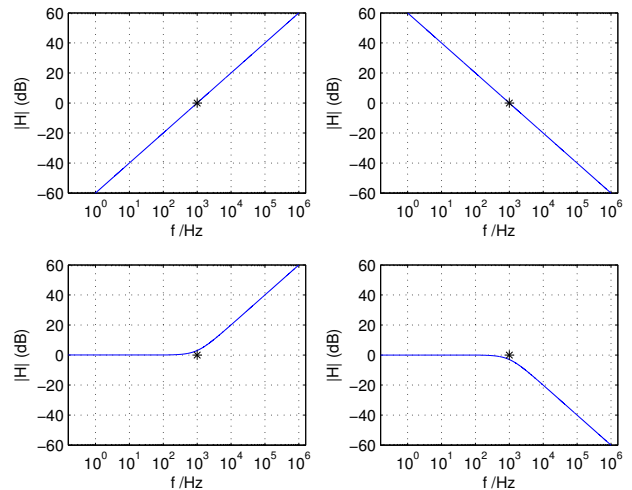
The phase plots are shown for the same terms, in the following figure, with a vertical line at the frequency corresponding to ω_0.
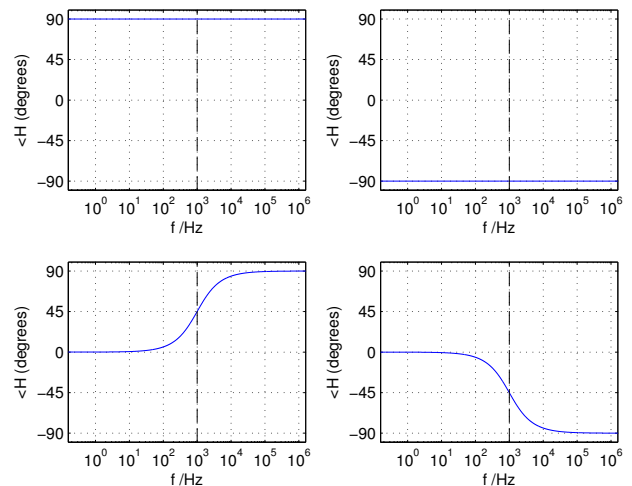
Given a function such as (1), each term can be identified and plotted *separately*, and then the resulting plots can be combined to give the complete network function, as shown in the following figure.
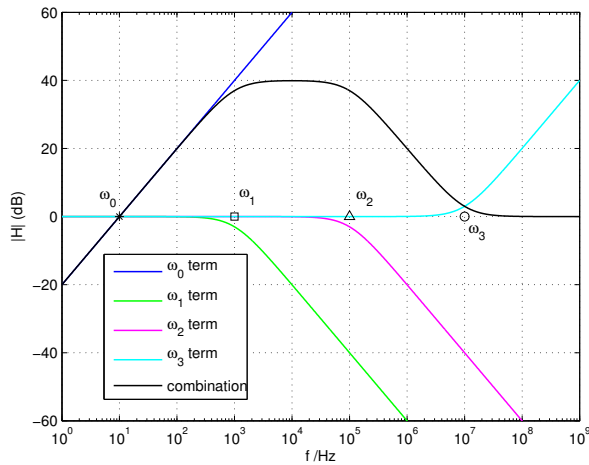
Notice that the decibel amplitudes of the separate terms have been added to find the decibel amplitude of the total network function. This may sound strange, as the separate terms are *multiplied* in the network function: it works because the decibel scale is logarithmic, and $\log ab = \log a + \log b$. Hence, for example, if $\mathrm{dB}(\cdot) = 20 \log_{10}(\cdot)$,

$$\mathrm{dB}\left(\frac{1 + \mathrm{j}\omega/\omega_1}{1 + \mathrm{j}\omega/\omega_2}\right) = \mathrm{dB}\left(1 + \mathrm{j}\omega/\omega_1\right) + \mathrm{dB}\left(\frac{1}{1 + \mathrm{j}\omega/\omega_2}\right).$$

For a phase-plot, the phases add because of the behaviour of complex numbers when multiplied: in polar form, $A\underline{/\alpha}\,B\underline{/\beta} = AB\underline{/\alpha + \beta}$.

The conventional Bode plot uses just straight lines, ignoring the curved part where a term such as $1 + \mathrm{j}\omega/\omega_x$ has the condition $\omega \simeq \omega_x$. The straight line plot is the 'asymptotic approximation' of the actual network function: hand-drawn plots generally use straight lines for simplicity. The frequency ω_x is called the *corner frequency* of such a term, since the slope changes between above and below this frequency.

7 A more detailed example

Our final laboratory task is Lab3, Kabelsökare. Its result is a circuit that will turn on little warning-lights when it detects that there's an energised conductor (with AC mains) close by. The circuit includes an opamp-based amplifier with a band-pass filter. It amplifies the signal picked up by a capacitive sensor that is held near the wire.
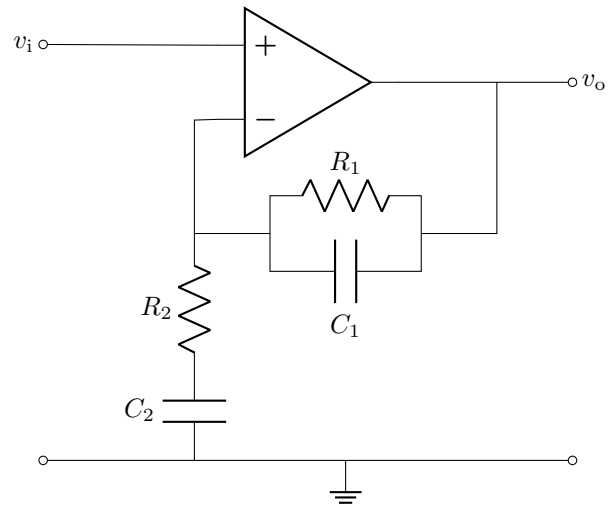
The input signal v_i is the potential of a small metal plate on the circuit board. If an energized wire is nearby, this function will contain a power frequency (50 Hz) part, picked up by capacitive coupling. It also will contain higher frequencies, due to noise in the supply, radio signals, etc. These signals may all be in the region of small parts of a volt, but there is also a dc voltage of around 4.5 V due to the design of the circuit (the input is in fact connected to the middle of a resistive divider).

At the output, shown by potential v_o, we want the 50 Hz signal: this is what is expected from the "hidden cables" that this cable finder is looking for. It needs to be made stronger (amplified) so that it can easily be compared to the level that we choose for making the lamps (LEDs) come on. But we certainly don't want to amplify the dc part of the input by lots of times: that would make a very high voltage, and our amplifier's output is anyway limited to the range $0 < v_\mathrm{o} < 9\,\mathrm{V}$ by the 9 V battery that drives the circuit. We also would prefer not to detect the frequencies much more than 50 Hz, which probably have come from other sources that we are not interested in detecting. For this reason we want an amplifier that gives the strongest gain at frequencies around 50 Hz, and low gain at frequencies far away from that: we want a *band-pass* filter, with its pass-band including 50 Hz.

For convenience, define an impedance to represent each R-C pair:

$$Z_1 = \frac{R_1 \frac{1}{\mathrm{j}\omega C_1}}{R_1 + \frac{1}{\mathrm{j}\omega C1}} = \frac{R_1}{1 + \mathrm{j}\omega C_1 R_1}$$

$$Z_2 = R_2 + \frac{1}{\mathrm{j}\omega C_2}$$

Voltage-division of v_o gives the potential v_- at the inverting input. Note that this voltage division is only valid because we know that the opamp input takes no current. The assumptions of negative feedback and an ideal opamp let us equate the potentials of inverting and non-inverting inputs.

$$v_\mathrm{i} = v_+ = v_- = \frac{Z_2}{Z_1 + Z_2}\, v_\mathrm{o}$$

This gives the network function for the whole amplifier circuit,

$$\frac{v_\mathrm{o}}{v_\mathrm{i}} = \frac{Z_1 + Z_2}{Z_2} = 1 + \frac{Z_1}{Z_2}$$

Now, unfortunately, we need to substitute the neat 'Z' symbols with the known quantities; it is also helpful to rearrange the result into the neat form with $\mathrm{j}\omega/\omega_x$ terms.

$$
\begin{aligned}
H(\omega) = \frac{v_\mathrm{o}}{v_\mathrm{i}} &\quad \cdots \\
&= 1 + \frac{\frac{R_1}{1+\mathrm{j}\omega C_1 R_1}}{R_2 + \frac{1}{\mathrm{j}\omega C_2}} \\
&= 1 + \frac{\mathrm{j}\omega C_2 R_1}{(1+\mathrm{j}\omega C_1 R_1)(1+\mathrm{j}\omega C_2 R_2)} \\
&= 1 + \frac{\mathrm{j}\omega/\omega_0}{(1+\mathrm{j}\omega/\omega_2)(1+\mathrm{j}\omega/\omega_1)}
\end{aligned}
\tag{6}
$$

where $\omega_0 = \frac{1}{R_1 C_2}$, $\omega_1 = \frac{1}{R_2 C_2}$, and $\omega_2 = \frac{1}{R_1 C_1}$.

In our case (the lab task), we have

$$
\begin{aligned}
R_1 &= 560\,\mathrm{k\Omega}, & C_1 &= 2.2\,\mathrm{nF}, \\
R_2 &= 12\,\mathrm{k\Omega}, & C_2 &= 1.0\,\mathrm{\mu F},
\end{aligned}
$$

whence

$$\omega_0/2\pi \simeq 0.3\,\mathrm{Hz}, \quad \omega_1/2\pi \simeq 13\,\mathrm{Hz}, \quad \omega_2/2\pi \simeq 130\,\mathrm{Hz}.$$

The following Octave/Matlab code can be used to plot the actual magnitude and phase response for this function, and the asymptotic approximation of the amplitude response.

```
% input
R1=560e3, R2=12e3, C1=2.2e-9, C2=1e-6,

% calculation
w0 = 1/(C2*R1), w1 = 1/(C2*R2), w2 = 1/(C1*R1),
f = logspace(-3,6,3000);  % vector of frequencies
w = 2*pi*f;  % corresponding angular frequencies
Z1 = (R1./(1j*w*C1))./(R1+1./(1j*w*C1));
Z2 = R2+1./(1j*w*C2);

% simple way to write H(w) [can check H_==H!]
%H_ = Z1./Z2 + 1;
% standard form of network function
% written with ".*" "./" to work on vectors:
H  = 1 + (1j*w/w0)./((1+1j*w/w1).*(1+1j*w/w2));
% H is now a vector of complex numbers giving
% the ratio vo / vi at different frequencies

% amplitude response
dB = 20*log10( abs(H) );
figure
semilogx( f, dB, 'b' );
xlabel('frequency [Hz]');
ylabel('amplitude response [dB]');

% phase response
figure;
semilogx( f, angle(H)*180/pi, 'b' );
xlabel('frequency [Hz]');
ylabel('phase response [degrees]');
```
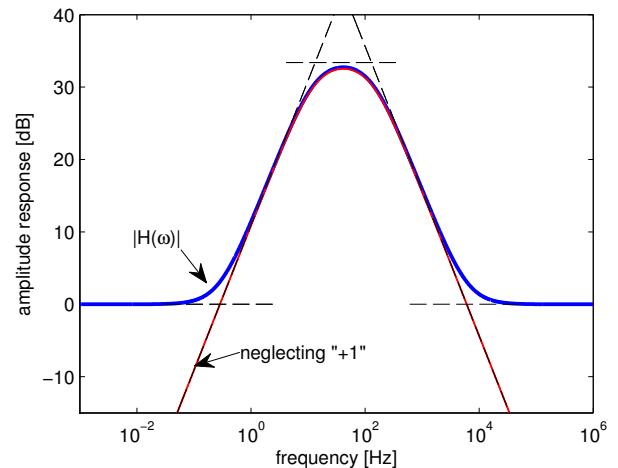
The resulting amplitude-response plot (after adding a little text to it) is the following:

The "+1" term ensures that the amplitude response cannot go below ×1, i.e. $0\,\mathrm{dB}$ (unless the real part of Z_1/Z_2 could be negative). This effect is seen by comparing the blue curve, for the entire function $H(\omega)$, with the red curve that shows this function without the +1 term. The aymptotic approximation (Bode-type plot) is shown too, by the dashed black lines; it makes clear the moderate deviation just at the special pole or zero frequencies, but good approximation over the rest of the curves.

The phase-response plot shows how the phase of Z_1/Z_2 (the "neglecting +1") changes from being dominated by the numerator's imaginary term at low frequencies, to being dominated by the one uncancelled pole at high frequencies. The full function $H(\omega)$ tends to zero phase at the extremes, as the 1 is then much bigger than the ratio Z_1/Z_2 so H is almost exactly real at very high or low frequency.
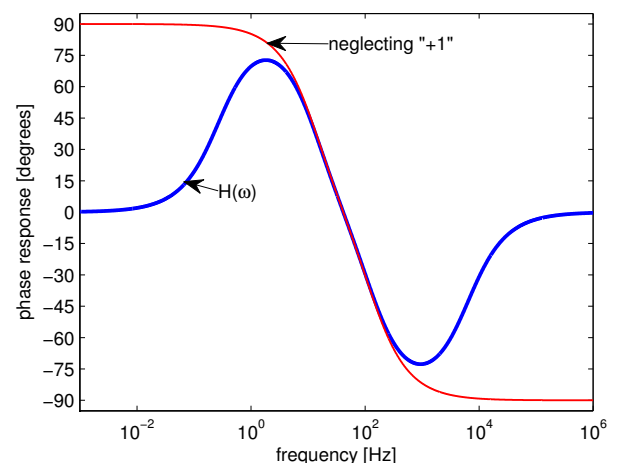
The following code is a more complicated alternative for the plotting, and was used to make the actual shown figures.

```
% (the variables used before are
%  assumed still to exist)
```

```
figure
% the first parts here make the asymptotic lines
semilogx( ...
    w0*[0.1,10]/2/pi, [0,0], 'k--', ...
    sqrt(w1*w2)*[0.1,10]/2/pi, ...
       [1,1]*20*log10(abs(w1/w0)), 'k--', ...
    (w1*w2/w0)*[0.1,10]/2/pi, [0,0], 'k--', ...
    f, 20*log10(abs(w/w0)), 'k--', ...
    f, 20*log10(abs(w1.*w2./(w0*w))), 'k--', ...
    f, dB, 'b', ... % plot the full function dB
    f, 20*log10(abs(H-1)), 'r' ...
);
set(gca,'ylim',[-15,45], 'xlim',10.^[-3,6]);
xlabel('frequency [Hz]');
ylabel('amplitude response [dB]');

% phase response
figure;  semilogx( ...
    f, angle(H)*180/pi, 'b', ... % full function
    f, angle(H-1)*180/pi, 'r' ...  % without "+1"
);
set(gca, ...
  'ylim',[-95,95], 'xlim',10.^[-3,6], ...
  'ytick',[-90:15:90] );
xlabel('frequency [Hz]');
ylabel('phase response [degrees]');
```
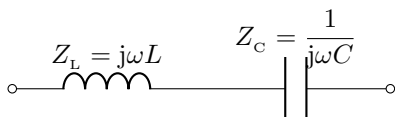
8 Second-order circuits: resonance

This subject has been touched upon in the exercises in the previous Topic, where the total equivalent impedance of series and parallel combinations are found. If a capacitor and inductor are connected together as a two-terminal combination, there is a frequency when they "cancel".

For example, for components L and C in series,

the equivalent impedance at angular frequency ω is

$$Z_{\text{eq}} = \text{j} \left(\omega L - \tfrac{1}{\omega C} \right)$$
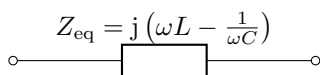
which is positive imaginary (like an inductor) when ω is large, negative imaginary (like a capacitor) when ω is small, and zero when $\omega L = \tfrac{1}{\omega C}$. The condition of $Z_{\text{eq}} = 0$ in the above circuit is *series resonance*. The frequency needed for series resonance can be found by rearranging the expression for $Z_{\text{eq}} = 0$, to

$$\omega = \frac{1}{\sqrt{LC}} \qquad \implies Z_{\text{eq}} = 0.$$

What is actually happening during series resonance is that a current passes through the components and causes a voltage in each, but the voltages are phase-shifted by 180°: in other words, they have opposite sign. The sum of the voltages is therefore

zero, meaning that the two components together appear to have no impedance (to *impede* is to hinder/resist/restrict). One special feature of this state is that there may be a large voltage across the capacitor and inductor separately, in spite of the voltage driving the circuit (at the terminals) being small. In this way a high voltage can be obtained from a low one: this is sometimes a useful thing to design for, but it is sometimes a problem when resonance is unintended.

If the components L and C are connected in parallel, the same condition exists for the resonant frequency, but the result of resonance is the opposite, as could be expected from duality:

$$\omega = \frac{1}{\sqrt{LC}} \quad \implies \quad Z_{\text{eq}} \to \infty, \text{ i.e.} \, Y_{\text{eq}} = 0.$$

With the parallel connection it is convenient to write the admittances, since these can be directly summed to find the total admittance,
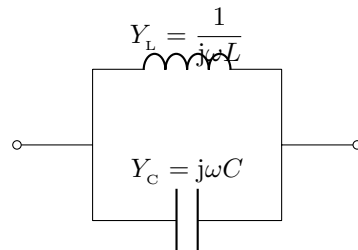
the equivalent admittance at angular frequency ω is

$$Y_{\text{eq}} = \text{j} \left(\omega C - \tfrac{1}{\omega L} \right)$$
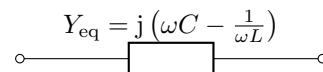
In parallel resonance the currents in the components may be large, but they are of equal size and 180° shifted. Therefore, no current moves through the terminals: the combination looks as if no current can flow.

[For this year we will not study second-order circuits as part of filters, except in the form of power-factor types of questions in later Topics. But this ought later to be extended to some basic 2nd-order circuit terminology about damping and Q-factors, etc.
Fill me in for next year!!]

9 — Extra —

9.1 Inideal R, L, C

In this course we've considered the basic components of resistors, capacitors and inductors to be ideal. That is a good choice, because the main intention is to learn how to solve ideal circuits. Then, the real circuits have to be translated into "ideal circuits that represent the real circuit with sufficient accuaracy"! This may often mean using several ideal components to model a single real component, or introducing limits.

One example we've already seen is how a typical voltage supply is better modelled by a Thevenin or Norton source with finite impedance, than by an ideal voltage source. We have also seen how real opamps have output limits, finite gain, and non-ideal input and output resistance. Some things that are commonly *modelled* as resistors, such as a light bulb (glödlampa) may be in fact a significantly non-linear resistor, as will be noted in the Topic on ac power.

But we haven't thought of the simple components of resistors, capacitors and inductors being nonideal. In the lab we had little cylinders with coloured stripes, which we modelled as ideal $10\,\mathrm{k\Omega}$ resistors, etc. How much, and how, do these deviate from ideal resistors.

Nonlinearity is fairly clear: we can reach some voltage or current where components melt or have flashover in the air around them, as well as more subtle nonlinearity such as the effect of heating. Another subtle nonideality, which exists for all components, is not of linearity but of frequency response.[6]

The frequency dependence of simple components comes from quite obvious causes, like the way that any set of conductors in the air have some finit capacitance, inductance and resistance.

The *thing* we call a resistor may be *intended* to be a pure resistance. But it involves a current going along a path, and therefore a magnetic field around the current: so there is necessarily some inductance in series with the resistance. It also involves a voltage drop between two the terminals of the resistor, resulting in an electric field between these points: so there is a capacitance in parallel with that part of the resistor. We could write a model made of three ideal components of L, C and R ... but that's still only approximate — for example, isn't there more than one plausible way to choose to connect these components into a two-terminal model of a real resistor?

Similar troubles happen for capacitors and inductors: they have resistance in their wires, inductors have capacitance between turns of their coils, capacitors have inductance in the internal electrodes, and some capacitors have significant conduction through the insulating (dielectric) material inside them. The word 'significant' is significant here, for in fact the amount that we would care about the nonideality depends on the frequency range we are interested in, as well as the accuracy that we need. If a capacitor is to be used at $400\,\mathrm{MHz}$ then a parallel resistance with a time-constant of $1\,\mathrm{s}$ can surely be ignored; but if used at $0.1\,\mathrm{Hz}$, or even $100\,\mathrm{Hz}$ if high accuracy is needed, it may be necessary to model the resistance as well. On the other hand, any series resistance will have a higher effect at the higher frequencies.

Lab components usually behave quite close to the ideal in the frequency range where we normally use them. At very high frequencies, the series inductance and parallel capacitance become important. (Think: e.g. inductive reactance increases with frequency, and series impedance is dominated by the biggest of the series impedances.)

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[6]It is a common mistake to describe a bendy line in a Bode plot as a nonlinear behaviour. That is a *frequency-dependent* behaviour, but not necessarily nonlinear. Linearity is about what happens if we make the entire stimulus (input) change in *size* by some factor, or add the effect of different inputs: it's about the scaling of voltage and current, not the frequency.