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Topic 11: AC Power

In the last two Topics we have rather hastily gone through some complex numbers, calculation of sinusoidal time-functions by the ac method (jω-method), and a little about filters and frequency response. All of that is quite general, or even a little biased towards the subjects of radio, audio, etc.

At this point, we move strongly towards an electric power bias, which is how we will stay for the rest of the course. First, we spend all of this Topic on just the concept of power in ac circuits (active, reactive, etc). The subsequent Topics will use a lot of the power concepts, applied in ways that are even more practically relevant to power systems.

Perhaps it sounds a little crazy to waste a whole Topic on *power* in ac systems — especially when the entire introduction to the ac analysis concept and complex numbers was only one Topic! We've already looked at how to find functions $u(t)$ and $i(t)$ describing the voltage across and current through any two-terminal device in a linear circuit in sinusoidal steady state, even where there are different frequencies driving the circuit. After calculating u and i for a component, the product is power,

$$p(t) = u(t)i(t), \tag{1}$$

where the direction of power-flow indicated by positive $p(t)$ is dependent on the directions in which voltage and current are defined. This equation is true at each instant in time.

In an ac circuit, the values of u and i are varying in time, so the power varies too. But over some time-span we can integrate the instantaneous power and find the energy transferred: we could do this for any time-functions. With the assumption of a periodic waveform, we can even calculate the mean (i.e. average) power in one period, and know that this result should be true for any period. The sinusoidal steady state is a nice easy case of a periodic waveform. So is that not enough?

Could we need to know more than this? Perhaps surprisingly, the answer is "yes!". There is no doubt about the meaning of instantaneous power, nor of mean (average) power over a given interval for a given time-signal. But there are lots of other definitions that are useful in different ways in different contexts. We shall look at some. Our soon-coming guest lecture is likely to include such words as *reaktiveffekt*, *effektfaktor*, *induktivlast*, etc., and to mention the importance of these concepts in power systems.

As long as we restrict ourselves to power in two-terminal devices in linear circuits in sinusoidal steady state conditions at a single frequency, then the subject is *quite* simple and well defined. We have that restriction in this course, although in the Extra section we look a little at exceptions.

1 AC power in time

This section analyses ac power by looking at time functions. It is intended to help the understanding of why the ac definitions are made the way they are, and what the subtle differences are between effective value and rms. It is not strictly needed for solving the normal exam tasks: the phasor-based methods in Section 4 are the main basis of ac power calculation.

We start by considering power in resistors. Loads (consumers of power) are often modelled as resistors. Back a few decades or a century ago, many loads — such as heaters and lamps — *were* well approximated as resistors: they *were* long thin pieces of metal that were heated by electric current passing through them.[1]

Nowadays, it's not so simple. Lamps are "low energy" lamps with power electronic converters on the input. Large heating systems may be heat-pumps, which are basically a power-electronic converter driving a motor.

Many other modern devices, such as computers, are also not very similar to resistors. If you halve the voltage to an ideal resistor, the power into it is quartered: remember $P = u^2/R$. In contrast, a modern computer power supply can do interesting things like demanding the same power even when the voltage is halved. That's because its power-electronic input is designed to work happily from 265 V down to 90 V, in order to work over the permitted ranges of voltage in the two main voltage-standards of the world, where many countries have around 230 V but North America and a few other regions have around 110 V. When the voltage is reduced, the device's controller causes more current to be used, in order to maintain the power input. These modern types of loads can also draw currents that are very far from being sinusoidal; see Section 8.4 for more detail.

[1] It is often assumed, in books on electric circuits, that a "lamp" (glödlampa) [link] is a linear resistor. But the filament's temperature when white-hot is about 3000 °C, and the metal (tungsten) used for the filament has a "temperature coefficient of resistance" of about 0.4% per kelvin (at room temperature). This coefficient means that at temperature θ the resistance will be $R_\theta = (1+4\times10^{-3}(\theta-20\,°\mathrm{C}))\cdot R_{20\,°\mathrm{C}}$. Clearly, if the temperature changes from about 20 °C to 3000 °C between zero current and normal rated current, then the resistance also changes by several times (although we should be careful about believing the equation very accurately: the temperature coefficient could change at the extreme temperatures). So a lamp is *not* well modelled as a resistor unless we know that we are only considering a narrow range of currents or voltages. When a lamp is turned on, it draws a significantly higher current in the first fraction of a second, until the filament has warmed up: failures (blowing) tend to happen at this time. But over shorter timescales such as 10 ms (the period of power-pulsation with 50 Hz ac current, as we will soon see) a filament's temperature does not change so much, so one can say that it is approximately a linear resistor within the ac cycle: the current will be approximately proportional to the voltage at each point during a period of 50 Hz steady-state sinusoidal excitation.

But these sorts of devices are not linear components. This course is about linear circuits. Also, practical approximate calculations about power systems are often based on linear circuits. So we continue with resistors!

1.1 DC power

Suppose we have a 3 V dc source, and want a resistance to consume a power $P = 6\,\text{W}$. Then it needs to draw a current of

$$I = \frac{P}{U} = \frac{6\,\text{W}}{3\,\text{V}} = 2\,\text{A},$$

and so the resistance is

$$R = \frac{U}{I} = \frac{3\,\text{V}}{2\,\text{A}} = 1.5\,\Omega.$$

We could instead have used the direct relation, that

$$P = \frac{U^2}{R}, \quad \therefore R = \frac{U^2}{P} = \frac{(3\,\text{V})^2}{6\,\text{W}} = 1.5\,\Omega.$$

With dc, the voltages and currents are constant in time, so the power by equation (1) is constant. This leads to the above simple relation between a resistor's voltage or current and the power dissipated in that resistor. It all seems very simple.

1.2 AC power in a resistor

Let us now consider the power into a resistor when a sinusoidal (ac) voltage or current is applied. In Section 8.2 there are some practical examples of when these fixed-voltage or fixed-current approximations are reasonable.

Consider taking the same resistor R as in the dc case above, but connected to a sinusoidal voltage source with peak value $\hat{U}$,

$$u(t) = \hat{U}\cos(\omega t).$$

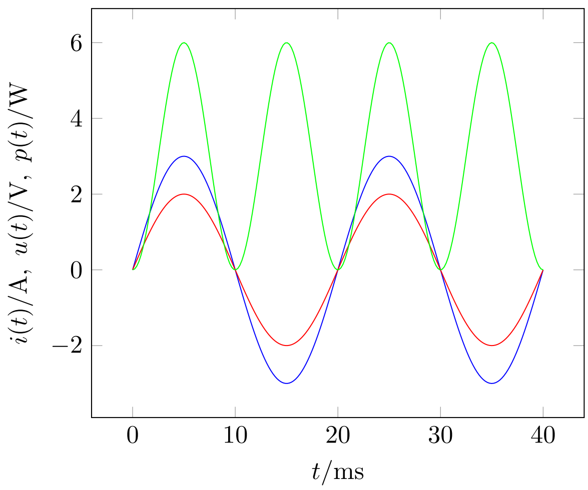
The current is then

$$i(t) = \frac{\hat{U}}{R}\cos(\omega t).$$

Multiplying these two quantities, we get a time-function for the instantaneous power (the power *in* to the resistor if the passive convention was used),

$$p(t) = u(t)i(t) = \frac{u(t)^2}{R} = \frac{\hat{U}^2}{R}\cos^2(\omega t). \tag{2}$$

The voltage, current and power are shown in the following figure, for the case where $\hat{U} = 3\,\text{V}$, $\hat{I} = 2\,\text{A}$, and $\omega = 2\pi \cdot 50\,\text{Hz}$. Their colours are blue, red and green, respectively.



There is nothing strange about the blue and red curves of $u(t)$ or $i(t)$, but the curve of $p(t)$ (green) may look surprising. It's always positive … which is reasonable, as a resistor cannot ever produce power. It alternates, but at *twice* the frequency of the voltage or current: after a little thought, that sounds reasonable, as there is still a positive power when the voltage and current are *both* negative. The shape looks like a sinusoid but added to a constant "dc offset".

The same thing can be handled in a symbolic, instead of pictorial, way. The time-dependence of the power was given by a $\cos^2(\omega t)$ term. A very useful relation for handling this $\cos^2$ and more general cases of multiplied cosines, which we come to later, is

$$\cos\alpha \cdot \cos\beta = \frac{1}{2}\big[\cos(\alpha+\beta) + \cos(\alpha-\beta)\big]. \tag{3}$$

Note that, as cosine is an even function, the second term is independent of the order of α and β.

Applying (3) to (2),

$$p(t) = \frac{\hat{U}^2}{R}\cos^2(\omega t) = \frac{\hat{U}^2}{2R}\big(\cos(2\omega t) + \cos(0)\big), \tag{4}$$

and because $\cos(0) = 1$, this is

$$p(t) = \frac{\hat{U}^2}{2R}\left(1 + \cos(2\omega t)\right), \tag{5}$$

which neatly describes exactly what we saw in the earlier graph: the instantaneous power is a double-frequency sinusoid, added to a constant value that is the same as the sinusoid's peak value; the sinusoid just touches zero at its lowest point.

1.3 Mean (average) AC power

Voltages and currents in an ac circuit are periodic. The mean power in any cycle is therefore the same. The power can be integrated over a single cycle, to find the energy per cycle; this energy can then be divided by the period to give the mean power. Suppose that the period of the voltage is T, i.e. $T = \frac{1}{f} = \frac{2\pi}{\omega}$. Then

$$P_{\text{mean}} = \frac{1}{T}\int_{t_0}^{t_0+T} \frac{u(t)^2}{R}\,\text{d}t. \tag{6}$$

Applying this to the above case of (5), the result is that

$$P_{\text{mean}} = \frac{\hat{U}^2}{2R} = \frac{1}{2} \cdot \frac{\hat{U}^2}{R} \tag{7}$$

This can be seen from the fact that $\cos(2\omega t)$ integrates to zero over any whole number of periods T.

The factor "$^1/_2$" has practical significance! Resistive types of load can generally be used with dc or ac voltage.[2] People doing calculations with ac circuits would appreciate not having to keep putting a $\frac{1}{2}$ into the power equations. It would be convenient to know that a 200 V ac or dc source will give the right power to a resistive load designed for "200 V".

There is a way to make the simple calculations such as i^2R and ui give the *mean* power, without having to write a factor of $\frac{1}{2}$. This is done by defining the magnitudes of voltage and current phasors by a number that is less than their peak value. Up to now, we have described sinusoidal time-functions by their *peak* value and phase, as well as an angular frequency that is the same for all phasors that we are solving in the ac analysis. The phasors represent steady-state sinusoidal quantities, and we chose to set the phasor magnitude equal to this peak value of the sinusoid. That just seemed an obvious choice. We've typically chosen a cosine reference for phase, and peak value for magnitude, and therefore have represented $\hat{U}\cos(\omega t+\phi)$ as a phasor $\hat{U}\underline{/\phi}$. But there is no requirement to use this choice of magnitude scaling and phase reference. We can choose something that is more useful in some way, for example by simplifying the power calculation.

2 Effective and RMS values

The normal way — within electric power engineering — of describing the magnitude of an ac voltage or current is *not* the peak value! Sinusoidal voltages and currents are instead specified by their "rms value", defined as $\hat{U}/\sqrt{2}$ or $\hat{I}/\sqrt{2}$; i.e. we define the phasor's magnitude to be $^1/_{\sqrt{2}}$ of the peak value of the sinusoid.

What happens when using this definition to calculate mean power into a resistor R? If we used the same equation as for dc power we would calculate

$$P = \frac{\text{voltage}^2}{R} = \frac{\left(\hat{U}/\sqrt{2}\right)^2}{R} = \frac{\hat{U}^2}{2R} = P_{\text{mean}} \tag{8}$$

By using this "rms" definition of magnitude, the simple dc equation gives us the correct mean power in a resistor in an ac circuit. Power calculations involve products ui or u^2 or i^2: there are always two circuit quantities multiplied. In a linear circuit, if we scale all the sources by some factor, then all the circuit quantities will be scaled by this factor. So if all currents and voltages have a scaling of $^1/_{\sqrt{2}}$ of their peak value, the factor 1/2 will always appear in power calculations, without having to be written.

[2] A bit more than a century ago, there were some dc and some ac supply systems. Then the advantages of ac, such as transformers, made ac the dominant choice. If a 100 W resistive load had been designed for a supply that was 200 V dc, it would have a resistance of 400 Ω. If it were then to be connected to a 200 V ac (peak) supply, it would only give 50 W (ignoring the temperature coefficient). An incandescent lamp running at half-power would be almost useless; it would give *much* less than 50% of its intended visible light. Users of equipment would probably like a "200 V supply" to run their 200 V equipment, without having to think about correction factors for dc and ac.

2.1 Effective value

[Not a core concept in the course.]

The term *effective value* (sv: *effektivvärde*) suggests choosing a number that describes "what can this really do" compared to some base case. Generally we compare to dc, as was done in the example above. Consider a non-dc voltage source, connected to supply power to something (not necessarily a resistor). If the source supplies, in a given time, the same energy as a dc source of U volts would have done, then we could say that the non-dc source has (over that time) an "effective value" of U. If the source is periodic, then the effective value over one cycle will of course be the same as for any other cycle or whole number of cycles. The same applies to the effective value of a current.

As was seen earlier, the effective value of a sinusoidal voltage applied to a resistor is $^1/_{\sqrt{2}}$ of the peak value. The same scaling would not necessarily be true for other waveforms and components. A sinusoidal voltage applied to a dc current source would have an effective value of zero for any whole number of cycles (show this, if you want). A square-wave current switching between $\pm\hat{I}$, applied to a resistor, would have an effective value of $\hat{I}$.

2.2 RMS value

The *root-mean-square* or *rms value* is a mathematical definition. But is can also be seen as a *particular type* of effective value, specific to resistors, which have a quadratic relation of instantaneous power to voltage or current.

The name means that one would square the time-function of voltage or current, then integrate to find the mean (average) of this squared value, then take the square root! For a periodic function $u(t)$ with period T, the rms value is thus

$$U_{\text{rms}} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} u(t)^2 \mathrm{d}t}. \tag{9}$$

This is basically what we did in (6) to find the mean power in a resistor, except that we didn't take the square root. The square root allows us to find a single (dc) voltage that would give the same mean power. You might see this more easily by squaring both sides: then you see that the calculation equates the time-means of the squares of the dc (left) and periodic (right) quantities. If a component has a quadratic dependence on voltage or current, then we see that the rms would be the effective value.

Other effective values, not the rms, would be appropriate to a device in which the power is *not* quadratically dependent on voltage or current. An extreme example is a "surge arrestor", which is a very nonlinear resistor. These are designed to permit large currents to flow when the voltage is higher than normal, which is good for passing lightning currents to ground without breaking down the insulation, but they should pass only a very small current when at the normal working voltages. In this case, something like $i \propto (u/u_0)^{12}$ may be the relation, where u_0 is a reference value. Then the "effective value" of a sinusoid would be closer to the peak value than the rms value is: my numerical calculation makes it about 0.89 instead of 0.71 (i.e. $1/\sqrt{2}$) of the peak.

2.3 RMS in sinusoidal conditions

The factor $1/\sqrt{2}$, seen earlier, is the ratio of the rms value to the peak value, specifically for sinusoidal signals. Try applying (9) to a sinusoid, to show this.

It is so conventional to quote ac voltages as rms values that we hardly think about it. The 230 V standard supply voltage that we have several times mentioned is an rms voltage: the peak is around 325 V. The same is true of the 400 kV between conductors of a HV transmission line: it is rms. Multimeters on an ac setting will usually show an rms value, or at any rate not a peak value.

2.4 Summary of RMS etc.

These three concepts of effective, rms and $1/\sqrt{2}$ are not exactly the same. Effective values are the most general: they need definition of a waveform and component. RMS values depend only on the waveform: the rms is the effective value for a resistor. The $1/\sqrt{2}$ factor is the rms for the particular case of a sinusoidal waveform.

I'm not aware of a different word in Swedish for rms: effektivvärde appears widely used in the sense of "rms". Few people care about the more general meaning of effective value: it is very commonly assumed that power dissipation is adequately modelled with linear resistors. It is also common to regard rms as synonymous with the number $1/\sqrt{2}$, as sinusoidal waveforms are commonly assumed. The terminology depends on the subjects that people mainly work with: for a lot of technical terms it seems one can't find any firm general definitions!

The above is for background knowledge, and to put the concept of the rms value into context. As far as *you* are concerned, for tasks about linear ac circuits in this course, the "rms" or "effective" or "peak/$\sqrt{2}$" are basically the same thing.

You need to be careful about when to use a factor $\frac{1}{2}$ for power calculations and when not to. You also need to be careful about whether phasors' magnitudes are defined by peak or rms value: this is important when calculating power *and* when converting phasors back into time-functions.

In summary of this section:

> When a sinusoidal voltage with peak value $\hat{U}$ or a current with peak value $\hat{I}$ is applied to a resistor, the mean power into the resistor is $\frac{1}{2}\hat{U}^2/R$ or $\frac{1}{2}\hat{I}^2 R$ respectively.
>
> When instead the sinusoidal voltage or current is described with its rms value of $U_{\mathrm{rms}} = \hat{U}/\sqrt{2}$ or $I_{\mathrm{rms}} = \hat{I}/\sqrt{2}$, the factor $\frac{1}{2}$ is not needed; the calculation is the same as for dc, U_{rms}^2/R or $I_{\mathrm{rms}}^2 R$ respectively.

3 Reactive components

All of the above was about resistors, which were easy to compare between the dc and ac situations. Two other components are important in ac circuits: the inductor and capacitor. They appear "parasitically" all over the place, as all power lines, cables and other equipment have magnetic fields and electric fields surrounding the currents and voltages. Many intentional capacitors and inductors are also used, to store energy in converters, or to cancel the effect of undesired capacitors and inductors.

3.1 Capacitor power, in time

So, let's do the same thinking as in (5), but with a capacitor instead of a resistor. Remember that a capacitor's current is

$$i(t) = C\frac{\mathrm{d}u(t)}{\mathrm{d}t},$$

so if

$$u(t) = \hat{U}\cos(\omega t)$$

then

$$i(t) = -\omega C\hat{U}\sin(\omega t) = \omega C\hat{U}\cos\left(\omega t + \tfrac{\pi}{2}\right),$$

and so

$$p(t) = \omega C\hat{U}^2 \cdot \cos(\omega t)\cdot\cos\left(\omega t + \tfrac{\pi}{2}\right),$$

which can be simplified to

$$p(t) = \omega C\hat{U}^2 \frac{1}{2}\left(\cos\left(2\omega t + \tfrac{\pi}{2}\right) + \cos\left(\tfrac{\pi}{2}\right)\right), \qquad (10)$$

where the last term is constant zero, as $\cos(\pi/2) = 0$,

$$p(t) = \omega C\hat{U}^2 \frac{1}{2}\cos\left(2\omega t + \tfrac{\pi}{2}\right). \qquad (11)$$

The instantaneous power into a capacitor or resistor have the similarity that both have an oscillating component at 2ω. But there is also a big difference: the capacitor has *no* constant component in the power, and its oscillation has a different phase-angle from the resistive case.

The pure sinusoidal function $\cos(2\omega t{+}\pi/2)$ shows that the mean power is zero. Energy moves in and out, but over a period (or even half a period) of the angular frequency ω there is no total energy transferred. There is a non-zero *current* and *voltage*, but these phasors are perpendicular (phase difference of $\pi/2$).

3.2 Inductor power, in time

For an inductor, a similar reasoning applies: from the relation $i(t) = \frac{1}{L}\int u(t)\mathrm{d}t$, the current is

$$i(t) = \frac{\hat{U}}{\omega L}\sin(\omega t) = \frac{\hat{U}}{\omega L}\cos\left(\omega t - \tfrac{\pi}{2}\right),$$

where the integration constant has been assumed to be zero, because of the assumption of sinusoidal steady state.

By the same calculation as was used for the capacitor, but with negative "$\pi/2$" terms, the power is

$$p(t) = \frac{\hat{U}^2}{\omega L}\frac{1}{2}\cos\left(2\omega t - \tfrac{\pi}{2}\right). \tag{12}$$

As with the capacitor, the current in the inductor is simply moving energy in and out, with zero mean.

3.3 Resonance and compensation

The only difference between equations (11) and (12) is that there is a phase-shift of π between the power oscillations. This gives us the idea that if we choose $\omega L = 1/(\omega C)$, meaning that both components have the same *impedance magnitude* at our angular frequency ω, then at any instant in time the power going in to the capacitor will equal the power coming out of the inductor (and vice versa).

This is just what was seen in the brief discussion of second-order filters in the earlier Topic on frequency-response. In that case, the network functions were of the main interest. In the case of power, we are usually interested in a particular frequency, where we want to cancel unnecessary currents. A little more about useful cancellation between inductors and capacitors is given in Section 8.3.

3.4 Combined resistive and reactive

When a resistor is combined with a reactive component, in series or parallel, we have an "in between" situation. The current and voltage do not have exactly the same phase angle, as they would be for a pure resistor. But they are not as far as $+\pi/2$ or $-\pi/2$ out of phase, as they would be for a pure capacitor or inductor.

We can split a sinusoidal current or voltage into a sum of sinusoids at different phase angles. When analysing a circuit where there is an arbitrary phase between a sinusoidal current and voltage, it is useful to use one of these quantities as a reference, and split the other into a part that is exactly in phase and a part that is exactly in quadrature ($\pm\pi/2$).

For example, we could split a current into a sinusoid that is in phase with the voltage, and another that is at plus or minus $\pi/2$. This would be very appropriate in a circuit with a resistor connected in parallel to an inductor or capacitor. The voltage is the same for both components so it is a good choice of reference. The part of the current in phase with the voltage would clearly be the current into the resistor, and the part in quadrature would be the current in the reactive component.

Instead, the voltage could have been split into parts in phase and at $\pm\pi/2$ from the current; this is well suited to a series circuit, where the part of the voltage in phase with the current is the voltage across the resistor.

In these mixed circuits, the mean power will be between 0 and $\hat{U}\hat{I}/2$, depending on the values of the resistor and the reactive component. This subject is studied with more calculations in Section 4, as phasors make it easier to find the in-phase and out-of-phase parts of current and voltage, compared to writing out sine and cosine functions.

4 AC Power with Phasors

4.1 RMS phasor magnitudes

In a linear circuit, if we scale all the independent sources by a constant factor, like $1/\sqrt{2}$, then all voltages and currents in the circuit will be scaled by that factor. This is a consequence of linearity.

It is therefore valid to use any arbitrary scaling that we feel like, between peak values of sinusoidal time-functions, and the magnitudes of the phasors that represent them for ac analysis.

It is necessary to use the *same* factors (backwards) if we want to convert from phasors to time-functions. It is also necessary to consider these factors if calculating power. The factor $1/\sqrt{2}$ is useful because it makes the power calculation simple. From (7), the mean power into a resistor due to an ac voltage U or current I, is just like the dc equations,

$$P_{\text{mean}} = \frac{U^2}{R} = I^2 R, \tag{13}$$

as long as we define the phasor U to be $\hat{U}/\sqrt{2}$, and I to be $\hat{I}/\sqrt{2}$, where $\hat{U}$ and $\hat{I}$ are peak values of the sinusoids.

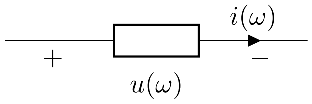
This ability to scale the magnitudes of phasors is just the same as we do for the angles. We choose any angle reference we like: often it is $\cos(\omega t)$ for convenience, but it could be e.g. $\sin(\omega t + 18°)$. The choice affects the angles of all the phasors; but they still keep the same relative angles, so the resulting equations give the same solution when translated back into time functions.

The choice of magnitude scaling and phase reference is arbitrary, but the choice can make the algebra much more or less simple!

Unless stated otherwise, **we will assume** that phasors used for power calculations in the following text have magnitudes that are the **rms values** of voltage and current. This is absolutely the usual assumption in power calculations.

4.2 Power: phasor-based S, P, Q, etc.

Let u and i be phasors with magnitudes that are rms values (sinusoid's peak/$\sqrt{2}$), defined on a two-terminal component. If the reference direction of i is into the reference positive side of u,



then the *complex power* (sv: *komplexeffekt*) into the component is defined as

$$S = ui^*, \tag{14}$$

where i^* is the complex conjugate of i.

The rectangular components of complex power are defined as $S = P + \mathrm{j}Q$.

The real part P is *active power* (sv: *aktiveffekt*), sometimes called real power. It describes the mean power delivered in each cycle of the periodic waveform.

The imaginary part Q is *reactive power* (sv: *reaktiveffekt*) or imaginary power. It describes, in the same scale as P, the components of current or voltage that are not in phase with each other and that therefore are only transferring energy in and out (no mean value). If S is defined *into* a component, then $Q > 0$ suggests an inductive impedance that can be modelled as inductance and resistance, $Q < 0$ suggests a capacitive impedance, and $Q = 0$ suggests pure resistance.

The reason for using the conjugate i^* when calculating S is that the power depends on the relative angles of voltage and current. The absolute angles mean nothing: they depend on our choice of angle reference. By choosing ui^*, we get $\angle S = \angle u - \angle i$, so S is purely real when u and i are in phase, regardless of what actual angle they have. The choice u^*i would also work, but the choice of ui^* makes Q be positive for an inductive load, which means that for the most common type of load (combined inductive and resistive) we can say that it "consumes active and reactive power". In our definition, we would say that a capacitor generates reactive power. This generated/consume distinction for reactive power is arbitrary, as both are just a zero-mean movement of energy.

The magnitude $|S| = \sqrt{P^2 + Q^2}$ is *apparent power* (sv: *skenbar effekt*)[3]. It's called that because it's what you might believe if you took a voltmeter and ammeter and measured the rms current and rms voltage then multiplied these real numbers: if you think in terms of dc circuits, this product would *appear* to give the power. Indeed, if the phase-angle between the ac current and voltage is zero, then this *does* give the mean power: $|S| = P = S$. But when the phase-angle is not zero (and not π either!), there is at least some component of the current that is in quadrature ($\pm\pi/2$) with the voltage. This represents an energy moving in and out: it increases the rms value, but does not contribute to the mean power.

The unit normally used for apparent power or complex power is the "volt-ampère" or "voltamp", with the symbol VA. The unit of P is the watt, W. The unit commonly used for reactive power is the reactive voltamp, which can be written as VAr or var. The different units help to distinguish between S, P and Q.[4]

The *power factor* (sv: *effekt faktor*), sometimes written as PF, is defined as PF $= P/|S|$. In view of the definition of apparent power (above), we see that power factor means "how much active power we actually got, compared to what we could have had with the same rms voltage and current if they were perfectly in phase with each other".
When $P = |S|$, then $Q = 0$ and PF$= 1$.
When $Q \neq 0$, PF< 1.

The PF does not numerically tell us whether $Q > 0$ or $Q < 0$; the Q^2 term in the calculation of $|S|$ ignores the negative sign. It is often of practical importance to know if Q is positive or negative. Two loads with apparent power 5 kVA and PF of 0.8 will become a combined load of 10 kVA and PF 0.8 if they have the same sign of Q. But if they have opposite signs of Q, they will form a combined load of 8 kVA at PF 1.0, i.e.

[3]Sometimes the apparent power is denoted just S, in which case complex power might be distinguished as **S**. We're being sloppy about not using special symbols for the complex numbers. Some people make them capital, while time-functions are lower-case. Some make them bold, or have overlines. That's quite useful if you want to use lots of absolute values: then the boldness can show if it's complex or real, instead of needing to write $|z|$ for each absolute part.

[4]If we look at the dimensions, in a simple way, all of S, P and Q are products of voltage and current, and could have the same unit of VA, which can be called a watt. But we prefer to use watts to describe only the active power; only the active power can be said to transfer "P joules per second" from one place to another. It doesn't make much sense to add S, P and Q directly, due to this difference in actual energy transfer. They are kept distinct by being real, imaginary and total parts of a complex number. An analogy in mechanics is the relation of torque (vridmoment), and energy exerted by a force: both can be seen as a force multiplied by a distance, but there is only really energy when the force and the distance are in the same direction. A more detailed dimensional analysis would consider the relative directions of the quantities.

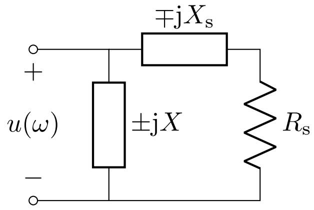
$S = P = 8\,\mathrm{kW}$, because their reactive currents cancel. So we add further information to the PF: we define it according to whether the current *lags* the voltage (inductive) or *leads* the voltage (capacitive). To lag or lead means, respectively, to come later (negative angle) or earlier (positive angle) in time. One talks of "a motor with PF= 0.9 lagging".

The phase-angle between current and voltage is the power-factor angle, θ. The power factor is then PF $= \cos\theta$. (You'll sometimes see power-factor marked on motors as the "$\cos\theta$" value.)

The neat relation $S = P + \mathrm{j}Q = ui^*$ gives everything we need about the relation of P and Q to S and to u and i. Lots of other relations can be expressed from the above, by algebra and trigonometry, such as $S = \frac{\hat{U}\hat{I}}{2}(\cos\theta + \mathrm{j}\sin\theta)$, but there's no point trying to remember them unless you find them useful. I can't see why you would find them useful: the calculation in rectangular form usually seems nicer. Just remember the ui^* definition, and practise handling some cases like $u^2/(R + \mathrm{j}\omega L)$ so that you become good at separating the real and imaginary parts.

4.3 Reactive compensation (PFC)

Power-factor correction is a common textbook-question. The classic task is a set of impedances that models a load connected through two terminals to a source, such as $\mp\mathrm{j}X_\mathrm{s}$ and R_s here:



A reactive component then needs to be connected at the two terminals, as for the component $\pm\mathrm{j}X$ above, in order that the load and reactive component together will look like a pure resistor.

Why? The practical purpoer is to minimise the current needed from the supply source, by allowing the load's reactive power to oscillate back and forward with the reactive power of the component that has been added. That way, only the current corresponding to active power has to flow in the wires from the supply source. See further applications of reactive compensation in Section 8.3.

Most loads, whether single motors or whole groups of houses, tend to be a bit "inductive": the current lags the voltage. So it is most common to need to connect a capacitor in order to supply the load's reactive power locally. But there are exceptions.

The main calculation difficulty is that you usually are *not* just told that an inductor and capacitor are in parallel and need to be made to have equal values. Instead — as with the circuit shown above — the load might be a series combination, but with the compensation connected in parallel. The correct value of the compensation must then be found from the reactance in the *parallel equivalent* of the load, which means a bit of fun with complex numbers. It might also be that you are asked to compensate only partially, e.g. so that the power factor becomes 0.9 instead of 0.7. It is often easiest to think in terms of comparing reactive power between the load and compensation, instead of comparing impedances or admittances.

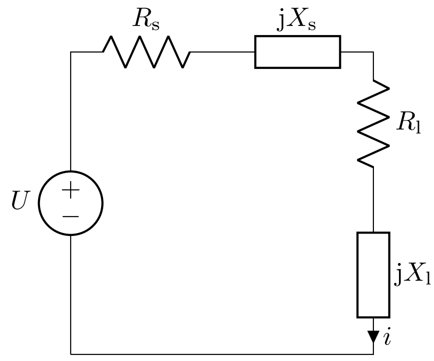
5 Maximum power: AC case

In dc circuits, maximum power transfer involved *choosing* a load resistor, for a *given* (fixed) Thevenin or Norton source, so as to get the maximum possible power dissipated in the load. The result (in Topic 04) was that this happens when the source and load resistance are equal. It was pointed out that it only makes sense to choose the load; the power transfer would always be increased by increasing the source voltage or reducing the source resistance.

In the ac case, maximum active power to a load is still a question of great practical interest ... perhaps greater! But there is the extra detail that the source and load can both be *impedances* that aren't necessarily purely resistive. There are therefore more "degrees of freedom" to consider, as we can adjust the load's resistance *and* its reactance. (It also can make some sense to talk about adjusting a source's reactance for maximum power for a given load.)

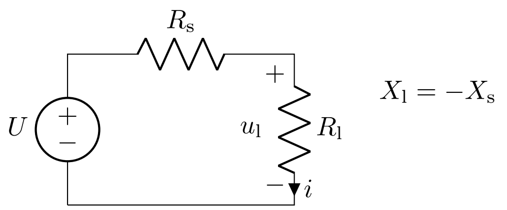
When active power in the load is to be maximised, we must maximise $P_\mathrm{l} = |i|^2 R_\mathrm{l}$; it is only the resistive part in the load that can consume active power. Therefore, we must maximise

$$P_\mathrm{l} = |i|^2 R_\mathrm{l} = \frac{|U|^2 R_\mathrm{l}}{|R_\mathrm{s}+R_\mathrm{l} \ + \ \mathrm{j}\,(X_\mathrm{s}+X_\mathrm{l})|^2}$$



Apart from the extra imaginary term in the denominator, and the need of taking absolute values ($|\cdots|$) this looks similar to a popular way of deriving the condition for dc maximum power.

Notice that X_l only comes into the denominator. Anything we can do that definitely reduces the *magnitude* of the denominator, while not changing the numerator, must help to increase the power. The imaginary part, $X_\mathrm{s}+X_\mathrm{l}$, should therefore be minimised in size: it always adds (at an angle $\pi/2$) to the real part, so it always increases the magnitude of the denominator regardless of whether it has a positive or negative imaginary value. A reactance contributes to the imaginary part of impedance in a positive way (inductor) or negative way (capacitor). So we can choose the load to have a reactance that exactly cancels the source reactance: $X_\mathrm{l}=-X_\mathrm{s}$. This is the best thing we can do with X_l to maximise power transfer, regardless of the value of R_l. After making this choice, the two reactances in the circuit above will cancel: they're a series resonance. If we used a Norton source and parallel load, this condition would still be true: the maximum power would come at parallel resonance between the source and load reactances, since this is when all the source current is available to go through the resistors.



The circuit looks then just like the dc maximum power situation, with a source and load resistor. We know that the maximum power in this case is with $R_\mathrm{l}=R_\mathrm{s}$.

The **maximum active power transfer to a load** Z_l in the ac case is thus achieved when

$$R_\mathrm{l}=R_\mathrm{s} \text{ and } X_\mathrm{l}=-X_\mathrm{s}, \quad \text{i.e.} \quad Z_\mathrm{l}=Z_\mathrm{s}^*. \tag{15}$$

Notice that the above paragraphs imply that the values of X are signed (e.g. so that it can be true that $X_\mathrm{l}=-X_\mathrm{s}$), which was convenient but is not the common definition we agreed on in Topic 09!

6 Power superposition

(This is not a very important part, exam-wise. It's just a possible method for simplifying a solution. It has some interesting practical implications. Put focus on the other parts.)

The Extra section of the previous Topic included some concepts about nonsinusoidal conditions, such as when there are nonlinear components in a system. It was pointed out that when a periodic voltage and current (on a two-terminal connection) contain various frequencies, then only the fourier components in voltage and current that have the *same* frequency *and* same phase can work together to produce power transfer: other combinations result in power simply moving in and out.

As a general case, consider time functions made up of a sum of cosines and sines at different frequencies,[5]

$$\begin{aligned} u(t) &= A_1\cos(\omega_1 t)+B_1\sin(\omega_1 t) \\ &+ A_2\cos(\omega_2 t)+B_2\sin(\omega_2 t)\ldots, \end{aligned}$$

$$\begin{aligned} i(t) &= a_1\cos(\omega_1 t)+b_1\sin(\omega_1 t) \\ &+ a_2\cos(\omega_2 t)+b_2\sin(\omega_2 t)\ldots, \end{aligned}$$

(In the special case where the voltage and current are periodic in time, with period T, their fourier series representations can be written in the above form: in that case, one ω_x value would be $2\pi/T$, and others would be integer multiples of it; a frequency of zero could be included to represent a nonzero mean value of the waveform.)

When the instantaneous power is calculated, the product of these time-functions leads to a set of products of cosines and sines,

$$\begin{aligned} p(t) &= u(t)i(t) = \\ & A_1a_1\cos^2(\omega_1 t)+B_1b_1\sin^2(\omega_1 t) \\ &+ (A_1b_1+a_1B_1)\cos(\omega_1 t)\sin(\omega_1 t) \\ &+ A_1b_2\cos(\omega_1 t)\sin(\omega_2 t) \\ &+ A_2b_1\sin(\omega_1 t)\cos(\omega_2 t) \\ &+ A_2a_2\cos^2(\omega_2 t)+B_2b_2\sin^2(\omega_2 t) \\ &+ (A_2b_2+a_2B_2)\cos(\omega_2 t)\sin(\omega_2 t) \\ &+ \ldots \end{aligned} \tag{16}$$

The only terms that will not have a zero integral over a long time are the ones with two cosines or two sines at the same frequency, i.e. A_1a_1, B_1b_1, A_2a_2, B_2b_2, etc. This can be seen by using the relations $\sin(\alpha)=\cos(\alpha-\pi/2)$ and $\cos(\alpha)\cdot\cos(\beta)=\frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$; when the multiplied frequencies are not equal, $\omega_x\neq\omega_y$, the result is made of sinusoids (at frequencies $\omega_x+\omega_y$ and $\omega_x-\omega_y$), and therefore has zero mean over a long time.

This feature of different frequencies leads us to a simplified solution method if we want to calculate the active power (mean power over "long enough" time[6]) in a circuit with multiple sources at different frequencies.

The *long* way to calculate power would be to do an ac analysis for each distinct frequency of the independent sources, and add these solutions by superposition to find the total current and voltage in the time domain: the product of these is then the instantaneous power,

[5] Note that a sinusoid of arbitrary magnitude and phase, $y(t)=C\cos(\omega t+\phi)$, can be described as a weighted sum of a pure cosine and sine function, $y(t)=A\cos(\omega t)+B\sin(\omega t)$.

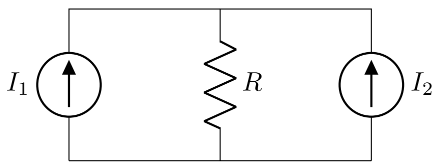
[6] "Long enough" to be a whole number of cycles of each frequency. When the frequencies are harmonics, i.e. integer multiples, then this is just one cycle of the *fundamental* (sv: *grundton*). In other cases it could take many cycles; a combination of frequencies 100 Hz and 101 Hz would need a time of 1 s. For practical purposes, just see it as being a long enough time to give a good approximation of the mean power.

which can be averaged over time to give the active power. That is very general, but takes some effort.

The *short* way, by observing that components of voltage and current at different frequencies will not lead to any mean power over a long time, is to calculate mean power from the ac analysis at each frequency separately, then sum them. In this way, we get all the $\cos^2$ and $\sin^2$ terms from (16), and these are the only ones contributing to mean power. The mean power (total active power) is then simply the active powers from the separate ac analyses; note that dc is just a special case of ac, where $\omega = 0$.

Superposition of powers due to sources acting separately would *not* be a valid approach for calculating powers in impedances due to sources at the same frequency. In that case the current in a component is still the sum of currents due to the separate sources, but the power that each current gives is determined by the voltages due to *all* the sources. To put it a shorter way: power in an impedance has a quadratic relation to voltage or current; it is not a linear relation ... so superposition is not applicable.

The following diagram is a powerful example!



Let $I_1 = -I_2 = I$, where I is an rms phasor (or a dc value). Calculate [active] power in R. The result by superposition of *power* (erroneous method) will be $|I|^2R$ from each source, therefore $2|I|^2R$. The result from correct calculation, by finding the current or voltage then finding the power from that, will be zero, as $I_1 = -I_2$ means that no current flows in R.

If instead the sources have different frequencies, then they cannot 'cancel' each other's currents in the way that happened above; they also cannot add to increase the total the whole time. Sometimes they will be cancelling and sometimes they will add. This is what was shown earlier with the multiplications of cosines and sines! In this case it is possible to add the powers due to the sources acting separately. *This is 'power superposition'.*

Even in the early days of electric power, some ingenious people understood this, and used one conductor as a common one for ac and dc circuits: that way, the power loss in the conductor only increased as the *sum* of what the separate losses would have been, instead of increasing as the *square* of the sum as would happen if two currents of the same frequency and phase were sharing the same conductor. [Hmm. Can I find the reference for people doing this? Or was it just a dream.]

7 Summary

[Normally I don't bother with summaries: I think that should be the reader's work, so as not to give the false impression of "this is all you really need to know, so don't read the other text". The past exams indicate which set of calculation skills is the main requirement in the course. But here's a reminder, with comments, since I wrote it last year anyway, and this Topic was a little heavy on definitions!]

All the following assume sinusoidal steady-state conditions!! That's the focus within ac power in this course. It's how almost all practical power calculations in power systems are done.

Complex power definition: $S = ui^*$, where u and i are phasors with magnitude of the rms values (sinusoid's peak/$\sqrt{2}$), Given passive convention, the above complex power is *in* to the component.

Name the rectangular parts as $S = P + \mathrm{j}Q$.
P is active (or real) power, describing the mean power delivered over a cycle.
Q is reactive (or imaginary) power, describing oscillatory flow that doesn't deliver mean power over a cycle.

Reactive power $Q > 0$ *into* an impedance means that the impedance has an inductive part; $Q < 0$ suggests a capacitive part; $Q = 0$ suggests purely resistive.

The magnitude $|S| = \sqrt{P^2 + Q^2}$ is apparent power, which is the product of rms voltage and current without caring about the relative angles. The real power might be smaller than or equal to the apparent power.

The power factor (PF) is P/S. When $Q = 0$, PF$= 1$. When $Q \neq 0$, PF< 1. When $Q > 0$ (defined into the load), the power-factor is "lagging" (the load consumes reactive power). When $Q < 0$ (defined into the load), the power-factor is "leading" (the load produces reactive power).

The phase-angle between current and voltage is the power-factor angle, θ. Then PF$= \cos\theta$.

Other definitions such as $S = \frac{\hat{U}\hat{I}}{2}(\cos\theta + \mathrm{j}\sin\theta)$ are obvious consequences of complex algebra and trigonometry. There is *no need* to try to remember or use them; try getting familiar with handling complex numbers instead. (Probably this comment isn't necessary now that we don't use the old book; that book seemed to encourage the above, which just leads the user into a mess with angles.)

8 — Extra —

8.1 Some relevance of ac power

Most of this Topic has been restricted to the case of power in linear two-terminal components, with voltages and currents being in sinusoidal steady state. This, for example, gave us the phasor-based expression $S = ui^*$, and quantities derived from this: P, Q, $|S|$ and $\cos\theta$ (PF).

It is common to assume these restrictions when considering ac power. This restricted situation includes the practically important case of reactive power in the inductances and capacitances of power lines and cables, and in inductive loads. In this case it is reasonable to see reactive power as a back and forward exchange of energy between different components: it causes voltage-drops and power losses in the long lines and cables, and restricts the available transfer of active (useful) power.

It is mainly in these sinusoidal contexts that network operators and policy-makers (government) consider things like "markets for reactive power services". These aspects of reactive power are highly significant for voltage control, line loading and transfer capacity in power systems from the lowest to the highest levels. Although the real ac power-systems are "3-phase" the analysis can be done similarly to the two-terminal case as long as the three phases are "balanced", which is usually a good approximation in normal conditions.

8.2 Practical voltage and current sources

This is further explanation of where it can be useful to consider resistors with a power u^2/R or i^2R.

In some practical cases, we know the voltage across a resistor. A power supply system usually behaves like a quite good approximation of a voltage source. In other words, there is not much variation in the supply's voltage when the power supplied by the source varies between the lowest and the highest intended values. The source is said to be a "stiff" voltage source, which could be modelled as a Thevenin source with low impedance. We expect that the voltage available from an outlet in a car will be close to 12 V (dc), and the voltage from an outlet in a building will be 230 V (ac). The lights might become a little dimmer when we connect a quite large load like a kettle or toaster, but the change in voltage should only be some percent. The electrical equipment we use is therefore designed for a particular voltage or range of voltage. At the correct voltage, these electric loads draw their designed current, to give whatever power is appropriate, e.g. < 0.1 A for a phone charger, but perhaps 10 A for a room heater on a 230 V supply. In these cases, calculations about power are naturally based on a fixed voltage applied to an impedance. A pure resistor can be used as a simple model of a load.

On the other hand, there is some relevance in resistors connected to current sources. The cables and transformers supplying power to loads carry a current that is almost independent of the cable's impedance. That's because the biggest impedance in the circuit is the loads. We've said that the voltage at a socket outlet should not change much when a new load is connected, and that this implies a low source-impedance seen at the socket. The source impedance is determined mainly by all those wires leading from the source of the supply. In order to get this small change in voltage with load, the cable impedance must be small compared to the load impedance. In that case, doubling the cable impedance does not make must difference to the current in the circuit, as the load impedance is the dominant part of the impedance in the series circuit of source, cable and load. Thus, "what the cable sees" is approximately a constant current source: a quite similar current flows in the cable, even for changes of tens of percent of the cable resistance. The power loss in the cable is of interest mainly in deciding whether the cable is thick enough that it will not overheat when at full load. For calculating this power loss, we can think of a resistor and a current source, where the power is $p(t) = i(t)^2R$ in the cable's resistance R.

8.3 Reactive compensation

It is the complementarity of inductors and capacitors, where their currents (in parallel) or voltages (in series) subtract from each other, that allows us to include one type of component in a circuit in order to "cancel" the effect of the other type.

If the capacitor and inductor have the same impedance and are in parallel, driven by a sinusoidal voltage source, a current will circulate between them, moving energy between the electric and magnetic fields. This is parallel resonance. The source will not have to supply any power *or* current. The two components will look like a zero admittance (infinite impedance). The energy oscillating between L and C was of course put there by something earlier, e.g. by the voltage source before steady state sinusoidal conditions were reached. Real components will have some resistance, so a small power is still needed in order for the oscillations not to die away.

If a capacitor and inductor with the same impedance are instead in series, driven by a sinusoidal current source, then the voltages across the two components will always cancel. The voltage on each component may be very large, but the sum is zero: the circuit then looks like a zero impedance.

Parallel connection is common with inductive loads, such as motors. These are often fitted with a parallel capacitor, in order to locally cancel the current that pointlessly moves energy in and out of the inductance; in this way the total current in the supply wires is reduced, so the power loss is reduced.

Capacitors or inductors can be connected in parallel with the power system in order to control the voltage. These are called “shunts”, as shunt is an old word for a parallel connection. They are controlled by switches or more finely by power-electronic switching. There are tens of places in Sweden where shunts can be connected to the power transmission system. Inductors cancel the effect of capacitance around the lines, when there isn't much power flowing. Capacitors cancel the voltage-change caused by large currents flowing through the lines' inductances.

Series connection has one application when the inductance of long transmission lines in the north and middle of Sweden is partially compensated by series capacitors at a few sites. This allows larger power transfers, which would otherwise be inhibited by problems with generators not being held together tightly enough by the network: the cancellation helps the Thevenin equivalent be a lower impedance.

Some electronic converters, such as for HVDC (the important dc connections common for sea-cables) generate harmonic currents that are too big to be accepted in the power system. These currents can be filtered by connecting a series LC combination that resonates at the harmonic's frequency. The harmonic current is “short-circuited” through the filter, while the current at the fundamental frequency (grundton) is limited by the capacitive reactance.

8.4 Nonlinear loads

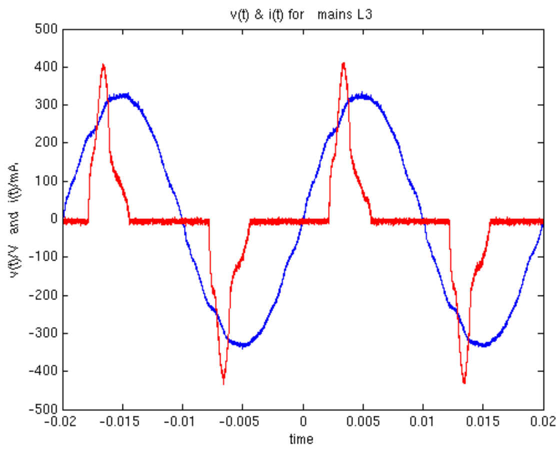
The situation becomes more complicated when the restrictions are removed. For example, a nonlinear load can take a non-sinusoidal current, even if the voltage is sinusoidal. The apparent power based on rms values of voltage and current will then be higher than the actual mean delivered power (active power). However, in this case there is not necessarily any storage of energy: we cannot just say that the difference between active and apparent power is explained by energy being stored in different places.

An explanation can be found by taking a fourier series of the voltage and the current: this assumes both are periodic, but not necessarily sinusoidal.

In power systems the voltage will usually be quite close to sinusoidal, as the network operator has to “provide” the voltage to customers (see `Energimarknadsinspektionen` on this!).

Small loads, however, do not have strong legal requirements on the waveform of their current, and the cheapest designs may draw *highly* non-sinusoidal currents. An example of strongly non-sinusoidal current (red) and approximately sinusoidal voltage (blue) is shown below.

This is from measurements made on cheap low-energy lamps, which have a simple diode-rectifier on their input. There are strong 3rd and 5th harmonics in the current. More current-waveforms, with calculated harmonic spectra, can be seen for some low-energy `Lamps` and some `ITequipment` such as computers and monitors.



Let's start by assuming that the voltage $u(t)$ is a pure sinusoid at the normal frequency f of the system, and that it is represented by a cosine in the fourier series: then all the other fourier series components (the sine term, and all the harmonics) will be zero. If a linear resistor were used as the load, its current would be directly proportional to the voltage, so it would also have a fourier series with just a single non-zero term: the cosine term at frequency f. If a linear load also contained some capacitance or inductance, then the current waveform would be sinusoidal but would have a phase-shift compared to the voltage: its fourier series would in general have a nonzero cosine and sine-term at f, but still no harmonics. The current into a *nonlinear* load will be non-sinusoidal (often called “distorted”); this means its fourier series will have some non-zero harmonics.

We know that the active power to a component is proportional to the time-integral of the instantaneous power over a period,

$$\int u(t)i(t)\mathrm{d}t.$$

The periodic waveforms can be written as sums of fourier-series components, e.g. $i(t) = A_0 + A_1\cos(\omega t) + B_1\sin(\omega t) + A_2\cos(2\omega t) + B_2\sin(2\omega t) + \ldots$.

If the voltage is simply $\hat{U}\cos(\omega t)$, then the active power will be $\frac{1}{T}\int_t^{t+T}(\,\hat{U}\cos(\omega t)A_0 + \hat{U}\cos(\omega t)A_1\cos(\omega t) + \hat{U}\cos(\omega t)B_1\sin(\omega t) + \ldots\,)\mathrm{d}t$ where T is the period of the periodic waveform. The interesting result from writing in this form is that we see that *only* one of these terms will *not* have a zero integral: this is the $\hat{U}\cos(\omega t)A_1\cos(\omega t)$ term. Products of cosine and sine, or between functions at different frequencies, have zero integral over a whole number of periods. You can see this from equation (3); it is a basic principle, on which indeed the calculation *of* a fourier-series is based!

From the above, we see that when the current has harmonics that the voltage does not have, then these parts of the current do not contribute to the energy delivered to a component. The harmonics do, however, increase the total current in the wires, and therefore the power *losses*. Harmonic currents also generate harmonic voltages by Ohm's law in the network's impedances. There are many further subtleties to the ways in which harmonics can even further increase the energy losses in transformers, motors, generators and wires.

8.5 Exchange

In polyphase (e.g. 3-phase) systems, or any other situation where there are more than two conductors to a load, it is possible for power to flow in through one pair of conductors and simultaneously out through another. This increases the current in the conductors and therefore increases the apparent power, but it doesn't increase the delivered active power. This is therefore another way, *without* needing energy storage or nonlinearity, that there can be a "reactive power", in the sense of a difference between the mean delivered power (active power) and the value that would be possible given the measured rms current (apparent power).