Electric Circuit Analysis, KTH EI1120

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Topic 12: Power and Transformers

The mutual inductors and [ideal] transformers can be seen as new components in this Topic, or as extensions of inductors. Transformers are often used in models of power circuits, and are often relevant to power calculations. Mutual inductance is a more general way to model transformers with more complexity. Both transformers and mutual inductors can potentially be replaced by equivalents made of already familiar components, in case you find it less scary to use dependent sources in your solution!

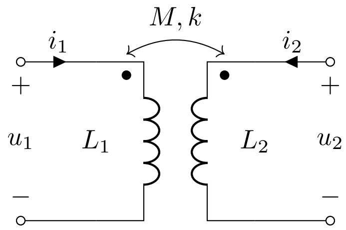
1 Mutual inductance

A single inductor is a construction where current between two terminals causes a magnetic field, and a change in this current (and therefore a change in the field) causes a voltage to be induced between the terminals. A deliberate inductor is typically a coil of wire, possibly with a magnetic core to increase the magnetic field. Undesired inductors are also formed by the single loops that all circuits consist of, such as tracks on a circuit board or power lines in the air.

What if we take an inductor, consisting of a coil of wire, and add another wire that follows a path very close to the first one? Then any voltage induced in the first wire should also be induced in the second, as they "feel" the same magnetic field. And if a current flows through this second wire, it will also contribute to the field. So currents in either wire affect the voltage induced in both! This magnetic coupling between two separate conductors is a *mutual inductance* (sv: *ömsesidig induktans*), for which we use the symbol M.

The two 'coils' do not have to be similar or very close: it is enough that any part of the magnetic field from one of them passes through the other. Nor do they have to be obvious coils with multiple turns: they could be two circuits working close to each other, such as adjacent power lines, adjacent tracks on a computer circuit board, or different coils in a motor or transformer. Practical situations often involve more than two magnetically coupled inductors: a mutual inductance can then be defined between each pair. We, however, consider just the basic case of two mutually coupled inductors. The term *coupled coils* (sv: *kopplade spolar*) is sometimes used for the general concept, even though the magnetically coupled circuits do not have to resemble coils. I think *coupled inductors* might be a better name.

A symbol for coupled coils is the following:



The circuit variables, as defined in this diagram, are related by the equations

$$\begin{aligned} u_1(t) &= L_1\frac{\mathrm{d}i_1(t)}{\mathrm{d}t} + M\frac{\mathrm{d}i_2(t)}{\mathrm{d}t} \\ u_2(t) &= L_2\frac{\mathrm{d}i_2(t)}{\mathrm{d}t} + M\frac{\mathrm{d}i_1(t)}{\mathrm{d}t} \end{aligned} \tag{1}$$

or for ac analysis,

$$\begin{aligned} u_1(\omega) &= \mathrm{j}\omega L_1 i_1(\omega) + \mathrm{j}\omega M i_2(\omega) \\ u_2(\omega) &= \mathrm{j}\omega L_2 i_2(\omega) + \mathrm{j}\omega M i_1(\omega). \end{aligned} \tag{2}$$

The coils look like normal inductor symbols. The familiar variables L_1 and L_2 can be called *self-inductances* if we want to distinguish them from the general case of inductance, which includes mutual inductance. They describe the contribution of $\frac{\mathrm{d}}{\mathrm{d}t}i(t)$ in a coil to the voltage induced in the *same* coil.

One of two new things in the above symbol is a two-headed arrow showing the coupling; in the above case it shows two component-values, M and k. The value M is the mutual inductance, describing the contribution of $\frac{\mathrm{d}}{\mathrm{d}t}i(t)$ in one coil to the voltage induced in the *other* coil. Note that this is the *same* value in both directions; we don't need one mutual inductance for the effect of coil 1 on coil 2, then a different one for the effect of coil 2 on coil 1.

The value k is called the *coupling coefficient*, and it describes how much of the magnetic field through one coil is also linked with the other. If $k = 1$, then all magnetic field through one coil also passes through the other. This is approximated when the coils are very close and perhaps even have a magnetic core. In this case, there is a relation between the three inductances: $M = \sqrt{L_1 L_2}$. If $k = 0$, then L_1 and L_2 are just two separate inductors with no coupling between them; we've already seen circuits with two separate inductors in earlier Topics! In the general case, with k ranging from 0 to 1,

$$M = k\sqrt{L_1 L_2}. \tag{3}$$

The other new thing in the symbol is the dots: one end of each coil has a dot. This is important for the direction in which the current in one coil affects the voltage in the other. The dots say "if the magnetic field linking the two coils is changed, then the voltage this induces in the coils is in the same direction in each, with respect to the dots".

For each one of the inductors, the choice of passive convention in defining the current $i_n(t)$ and voltage

$u_n(t)$ will ensure that a positive sign is needed for the contribution of $L_n \frac{\mathrm{d}}{\mathrm{d}t} i_n(t)$ to $u_n(t)$. If a coil's voltage is defined in the same way as the other coil's voltage, with respect to the dots, then the mutual inductance term will have the same sign as that other coil's self-inductance term: the following paragraph gives examples.

The situation shown in the above diagram is that passive convention is followed on both coils, and the voltages have the same definition with respect to the dots.[1] This leads to the equations (1) and (2). If both dots were swapped (put at the bottom) the equations would stay the same. If just one dot were swapped, then both mutual inductance terms ($M \frac{\mathrm{d}}{\mathrm{d}t} i_x$) would be negative. If both coils had active convention, and their voltages were defined in the same way with respect to the dots, then all four terms in the equations would be negative.

A good starting point for solving an ac circuit with mutual inductance is simply to write the circuit equations for the rest of the circuit in terms of u_1, u_2, i_1, i_2, then add the equations for the mutual inductor, (2). However, if you don't like the symbol, you could notice that (2) are the same as could be obtained by replacing the "M,k" part of the mutual inductor with a pair of current-controlled voltage sources! On the left side, a CCVS would be in series with the inductor L_1, and would have a value of $\mathrm{j}M$ and controlling current i_2; a symmetric situation would be done on the right-hand side. This could help to get used to the concept of what a mutual inductance does; but I suspect it's not very useful in the long term.

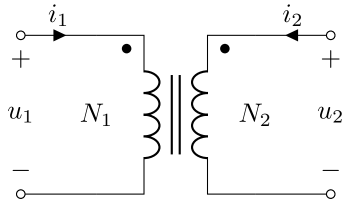
2 Ideal transformers

The "ideal transformer" is a special case of mutual inductance, where $k = 1$ and $L_1 \to \infty$ and $L_2 \to \infty$ (and therefore $M \to \infty$).

A high value of k is achieved by having windings close to each other, usually around a common magnetic core. From a practical viewpoint, inductances are "infinite" if the amount of current they would draw when connected to a voltage source ($u/(\omega L)$) is very small compared to the current we expect to have when both coils are in use.

The symbol for an ideal transformer typically includes two lines to show an "iron core", which could even be included for magnetic-cored mutual inductors in general. We can also write "ideal" to make it explicit.

[1] And, if the voltage are defined in the same direction with respect to the dots, *and* the same convention is used on each coil for the relative voltage and current directions of that coil, then the currents also will have the same definition with respect to the dots.



ideal

The biggest difference from the general coupled coils is that the inductances and coupling are not included: they are already defined above, as being as big as possible. Only one thing is important: the ratio of the number of turns on the two coils, which can be written $r = N_2/N_1$ for the above case.

As the coils are perfectly coupled ($k = 1$), all magnetic field through one of them also passes through the other. Any change in this field therefore induces the same voltage in every *turn* (sv: *varv*) of each coil. This can be expressed as

$$\frac{u_1}{N_1} = \frac{u_2}{N_2}$$

which gives the well-known transformer equation for voltages,

$$\frac{u_2}{u_1} = \frac{N_2}{N_1} = r. \tag{4}$$

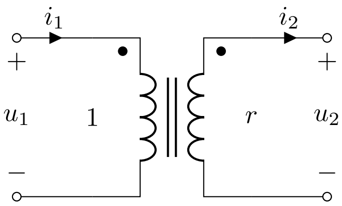
A two-winding transformer usually has one side referred to as the *primary*, and the other as the *secondary*. Sometimes the higher voltage one is called the primary, or else the one that is expected to have the power input might be called primary ... or just an arbitrary choice.

The very high inductances imply that even a very small current can cause a large amount of magnetic field linking with the coils. Even a small rate of change of current will lead to a very large induced voltage. This means that the rate of change of the total current passing around the magnetic core must be very close to zero: otherwise, extreme voltages would be induced, in a direction to oppose the change of total current that is causing them (Lenz's law). By the "total current" we mean that if current i_1 goes around the core N_1 times and therefore is $i_1 N_1$, and current i_2 (defined in the same direction with respect to the dots) goes around the same way N_2 times, then the total going around the core in a particular direction is $i_1 N_1 + i_2 N_2$. If the currents and voltages are ac, as is often the case when using practical transformers, we could say that the rate of change[2] of total current is $\omega(i_1 N_1 + i_2 N_2)$. This suggests that, in the ideal transformer, the phasors i_1 and i_2 are related by $i_1 N_1 = -i_2 N_2$, so that zero total

[2] If a current is sinusoidal with rms (or peak) value i at angular frequency ω, then its time-derivative is sinusoidal with rms (or peak) value ωi; if we care about the phase angle we should also add a shift of $\pi/2$ (multiply by j) since $\frac{\mathrm{d}}{\mathrm{d}t}\cos(\omega t)$ is $\omega\cos(\omega t + \pi/2)$.

current goes around the core: the two currents must cancel each other.

However, a more normal way to define the currents in a transformer is to use the active convention on the "secondary side". The secondary is the coil numbered 2, and the definition is usually by power flow. The power usually comes in to the primary side, and leaves at the secondary. This is why the passive and active conventions, respectively, are sensible. There are situations where the power flow can be different at different times; so the terms primary and secondary, and the choice of current definitions, can be defined various ways. The following diagram shows a transformer where the ratio of N_2/N_1 is simplified to r, and the secondary current is defined by the active convention.



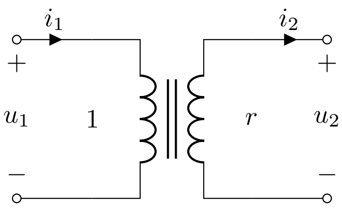
ideal

With this definition of current, the relation of the currents at the two sides becomes $i_1 N_1 = i_2 N_2$, so

$$\frac{i_2}{i_1} = \frac{1}{r} = \frac{N_1}{N_2}. \tag{5}$$

This shows that the currents are changed by the turns ratio r in the opposite way to the voltage. The result can be expected if one considers that the ideal transformer, with no resistance or stored magnetic energy, must always have the same instantaneous power in and out: the currents must therefore scale the opposite way to the voltages, as $p(t) = u(t)i(t)$.

In a sloppy case people might not bother to write dots, nor to write "ideal" explicitly. In many practical applications the circuits on the two sides are not connected to each other except possibly by a ground point, and the variables of interest are voltage magnitudes and powers but not relative phase-angles between the two sides. In this case the dots are not needed.



2.1 Circuit solutions

To model an ideal transformer in a circuit, we can follow the same idea as with coupled coils in general: write out equations for the circuit(s) around the coils, including the 4 defined variables of u_1, u_2, i_1, i_2. Then add the equations (4) and (5) that the transformer imposes between u_1 and u_2, and i_1 and i_2. The resulting equation system can then be solved, if the rest of the circuit is sufficiently well defined. However, the rigid ratios of voltage and current make the ideal transformer an easier case to solve than general coupled coils.

If some information in a transformer circuit allows you to know the current or voltage in one side of a transformer, then you can fill the corresponding current or voltage on the other side, scaled in the appropriate way by N_1 and N_2. If you don't directly know a current or voltage at the transformer, but have equations to relate it to other components, then you can add the transforer equations to your list of known equations, and do some substitutions.

One common method of simplifying a transformer circuit is to translate an impedance across the transformer. That means that an impedance at one side of the transformer is replaced by an impedance at the other side of the transformer, chosen in a way that is equivalent when seen from a part of the circuit that you are studying. A simple case is a resistor R connected to an N_2-turn secondary, while a voltage source U is connected to an N_1-turn primary. The resistor sees a voltage $\frac{N_2}{N_1}U$, so its current is $i_2 = \frac{N_2}{N_1}\frac{U}{R}$. The current on the primary side is $i_1 = \frac{N_2}{N_1} i_2$, which is $i_1 = \frac{N_2}{N_1}\frac{N_2}{N_1}\frac{U}{R}$. If instead a resistor defined as $R' = \frac{N_1^2}{N_2^2}R$ were connected to the primary, it would behave in the same way as the resistor R on the other side of a N_2/N_1 transformer. Hence, impedances can be moved between sides of an ideal transfomer, when scaled by the square of the turns ratio.

More generally, a Thevenin (or Norton) source coupled through a transformer can be modelled by scaling its impedace as above, and its voltage by the appropriate transformer ratio.

3 — Extra —

3.1 Links

The Wikipedia page on `transformers` is a good place for some further details and nice diagrams and pictures.

3.2 Why ac ?

Some very early power supplies, for local lighting, were dc, at tens or hundreds of volts. That was around the 1880s. One advantage of dc was that batteries ("accumulators") could be used for storage. Another was that dc motors were quite well known. It was presumably easier to operate multiple dc generators in parallel, as there would not be a need to get them to a similar speed and angle (phase) before connecting them. For small, local systems dc was a reasonable choice.

One major advantage of ac is that transformers can be used to change the voltage. An *ideal* transformer (with its infinite self-inductance of each winding) could transform dc. But that is just a curiosity of the imagination: a practical transformer definitely cannot do so! If a dc voltage is applied to one winding, the magnetic field in the transformer must keep rising in order to provide an induced voltage equal to the applied voltage. That means the input current must keep increasing — remember how inductors behave in dc steady state ... a short-circuit! The increasing magnetic field in the transformer causes the magnetic material (iron) used for the core of a practical transformer to *saturate* (sv: *mättas*), after which the current will have to increase many times more quickly, to maintain a given increase of magnetic field. In more practical terms, a transformer designed for 50 Hz might demand only 1 A or less to magnetize it at the correct voltage and frequency, but after a few tens of milliseconds of a similar voltage of dc being applied the transformer would draw a much higher current, looking almost like a short-circuit.

Conversions between ac and dc or between different voltage levels of dc are made possible nowadays by power electronics. Using inductors, capacitors, and electronic switching devices that rapidly move energy between these reactive components, conversion can be done with high efficiency and high controllability. Sometimes a transformer is included in the converter: in that case it is usually run at a much higher frequency than 50 Hz, made possible by the switching devices, in order to permit it to be a small transformer.

Conversions of dc/dc or ac/dc *were* also possible back in those early days of electricity supply, using the obvious method of a *motor-generator set* — this is where a motor drives a generator through a mechanical connection. However, a motor-generator combination also will have higher size, cost, noise and losses, and lower reliability, compared to transformers.

A significant problem with rotating machines (motors and generators) is that they are not easily made for very high voltages. Even today, the biggest generators only work at up to 20 kV or 30 kV, then their output goes through a transformer into the electric transmission system at 400 kV. The high voltages are highly desirable for efficient transmission of electricity.

Transformers are stationary and simpler to build; they are also relatively easy to make for high voltages. The main restriction is that they require a voltage *without* any significant dc. When transformers can be used, they can "step up" the voltage from generators. Then the electricity can be transferred efficiently over long distances, by using high voltage and low current, thereby reducing the power lost in the resistance of the wires. The voltage can then be reduced by other transformers to reasonable levels for using inside buildings.

It is also easier to make switches and circuit-breakers for ac, as the constantly changing direction of the current enables the switch to stop the current near the point when it is naturally zero in the ac cycle. Generators and motors can be built in larger sizes for ac, since the windings where the main power is generated or consumed can be stationary and directly connected to the power cables. In contrast, practical dc machines contained some form of sliding contact through which the main current had to flow, because the current generated in the windings is naturally ac. The 'alternator' found in a combustion-engined car is a three-phase ac machine, with a set of diodes built in to it to provide a dc output suitable for the battery and car electrics; this is another way to get dc from an internally ac machine.

3.3 Models for nonidealities

A common way to handle nonideal components is to model them with ideal components. One example is a real voltage source, which can be modelled to a better approximation as a Thevenin source in which and ideal impedance is limiting the output of an ideal voltage source.

An ideal transformer can be added to in this sort of way, to make a more detailed model of a practical transformer. Series resistance models the resistance of the metal wires. Series inductance models the part of self-inductance of a coil that is not included in the mutual inductance: i.e. the effect of flux that couples only one coil. Parallel inductance models the need of a finite current to force a magnetic field in the core, and a parallel resistance can model the power due to heat-losses in energising the core.

These models are popular as they allow idealised components to be used. Many people in electric power engineering appear to prefer models based on an ideal transformer surrounded by some other ideal components to model a transformer's nonidealities, instead of using a model based on mutual inductance.