# Tentamen: EI1102/EI1100 Elkretsanalys, 2014-01-15 kl 14–19

**Hjälpmedel:** *Ett* A4-ark med studentens anteckningar (båda sidor). Dessutom, pennor!

Svar får anges på svenska eller engelska. En kort ordlista finns på baksidan.
Tentan har 3 tal i del A (15p), och 3 tal i del B (15p).
Godkänt vid ≥25% på del A och del B individuellt, *och* ≥50% på delarna A och B tillsammans. Betyget räknas från summan av A och B. Eventuella bonuspoäng från KS och hemuppgifter tillkommer enligt KursPM. Se också PM:et angående betygsgränser, rättningsnormer och överklagande.

Läs varje tal noggrant **innan du försöker svara**.
Tänk på att **använda återstående tid till att kolla igenom varje svar**: man kan göra dimensionsanalys, rimlighetsbedömning (t.ex. "är det rätt att $y$ går ner medan $x$ går ner?"), och lösning genom en alternativ metod. Lösningar ska **förenklas** om inte annat är specificerat.
**Satsa inte för mycket tid** på bara en uppgift om du fastnar: ta hänsyn till poängvärden på uppgifterna, och att man måste både delar av tentan. Det är ofta så att **senare** *del*tal är betydligt **svårare** än de första deltalen.

Examinator: Nathaniel Taylor

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## Del A. Likström och Transienter.

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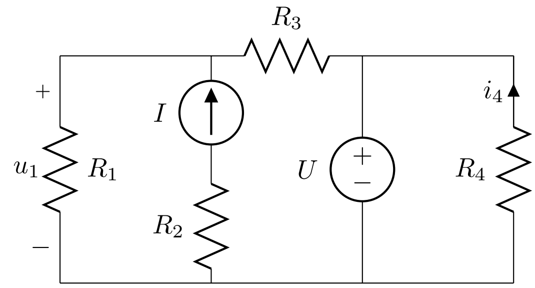
**1)** [5p]
Kända kvantiteter: $R_1$, $R_2$, $R_3$, $R_4$, $I$, $U$.




**a)** [2p]
Bestäm strömmen $i_4$.

**b)** [3p]
Bestäm spänningen $u_1$. Dimensionskontrollera resultatet.

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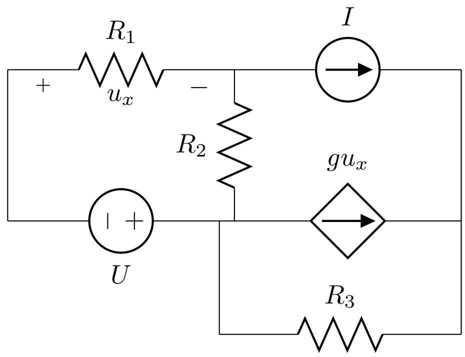
**2)** [5p]
Kända kvantiteter: $R_1$, $R_2$, $R_3$, $U$, $I$, $g$.




Definiera en jordnod, och potentialer för andra noder. Använd nodanalys för att skriva ekvationer som kan lösas för att få ut nodpotentialer som funktioner av de kända kvantiteterna.
Du *måste inte lösa* ekvationerna, och *måste inte* skriva om dem i förenklad eller matris form. Men de måste innehålla all information som behövs för att lösa ekvationerna för alla nodpotentialerna. Därför måste det finnas lika många okända variabler som oberoende ekvationer. Det finns flera möjliga svar (alla med samma lösning).

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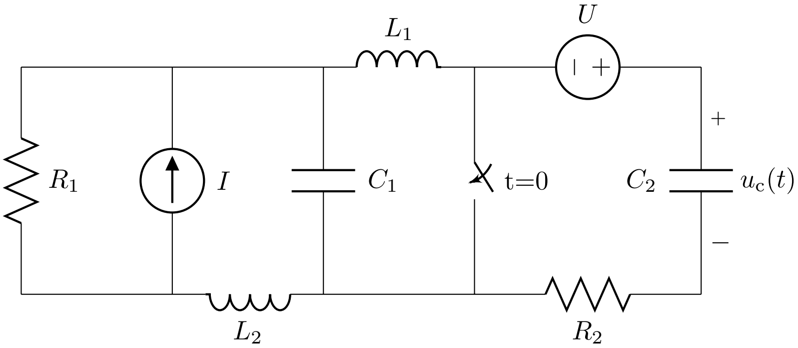
**3)** [5p]
Alla märkta kvantiteter *förutom* $u_c(t)$ är kända och konstanta över all tid.
Brytaren stängs (går från öppenkrets till kortslutning) vid tid $t = 0$.

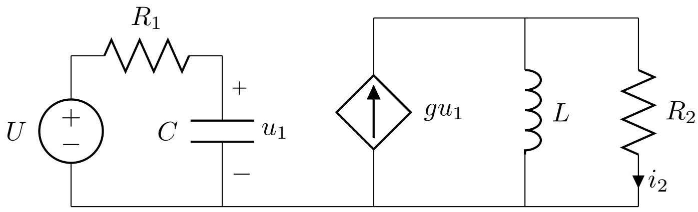



**a)** [2p]
Bestäm $u_c(0^-)$, d.v.s. $u_c(t)$ direkt *innan* brytaren stängs.

**b)** [3p]
Bestäm funktionen $u_c(t)$ för alla tider $t \geq 0$.
Observera att kretsen till vänster av brytaren är relevant bara med hänsyn till begynnelsevärdet.

## Del B. Växelström

**4)** [5p]
Kända kvantiteter: $R_1$, $C$, $g$, $L$, $R_2$.
$U$ visar amplitud och fas av en sinusformad spänning som funktion av vinkelfrekvens $\omega$.




**a)** [2p]
Bestäm nätverksfunktionen $H(\omega) = \dfrac{i_2(\omega)}{U(\omega)}$, uttryckt i de kända kvantiteterna.

**b)** [1p]
Visa att svaret till del 'a)' kan skrivas i formen $H(\omega) = \dfrac{k\,\mathrm{j}\omega}{(1+\mathrm{j}\omega/\omega_1)(1+\mathrm{j}\omega/\omega_2)}$.

**c)** [2p]
Skissa ett Bode amplituddiagram av funktionen $H$ från del 'b)'.
Anta att $\omega_1 \ll \omega_2$, och att $k = 1/\omega_1$. Markera viktiga punkter och lutningar.

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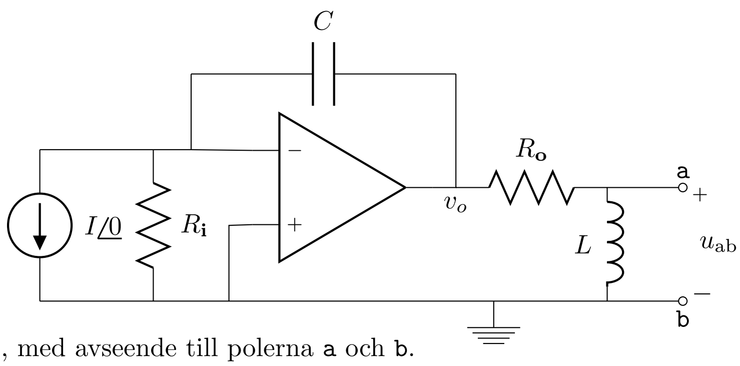
**5)** [5p]
Strömkällan är en växelströmskälla med vinkelfrekvens $\omega$ och amplitud $I$.
Kända kvantiteter: $I$, $\omega$, $R_\mathrm{i}$, $C$, $R_\mathrm{o}$, $L$.
Operationsförstärkaren är ideal.




**a)** [2p]
Bestäm $v_o$.

**b)** [2p]
Bestäm Theveninekvivalenten av denna krets, med avseende till polerna `a` och `b`.

**c)** [1p]
En extern impedans $Z$ ska anslutas mellan `a` och `b`.
Vilket värde av $Z$ måste väljas för att maximera effekten (reell effekt) som överförs till impedansen?

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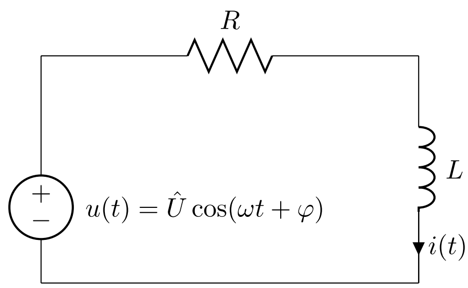
**6)** [5p]
Kände: $\hat{U}$, $\omega$, $\varphi$, $R$, $L$.




**a)** [2p]
Uppgift: bestäm $i(t)$ genom j$\omega$-metoden.

**b)** [2p]
Bestäm den komplexa effekten $S = P + \mathrm{j}Q$ som levereras från spänningskällan.
(En del från lösningen till 'a)' kan vara användbart här, då komplexeffekt är ett koncept inom j$\omega$-metoden.)

**c)** [1p]
Spolen $L$ tas bort. Två kopplade spolar tas in i kretsen. En av dessa spolar har självinduktans $L_1$, och kopplas i kretsen i platsen av den borttagna $L$. Den andra av de kopplade spolarna har självinduktans $L_2$, och är inte ansluten till någonting (öppenkrets). Kopplingskoefficienten mellan spolarna är $k$.
Bestäm amplituden (storleken) av spänningen över den andra spolen.

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**Ordlista över mindre självklara översättningar:** source *källa*, power *effekt*, switch *brytare*, terminal *pol*, (ground-)node *(jord)nod*, short-circuit *kortslutning*, coil/inductor *spole*, initial value/condition *begynnelsevärde*, opamp (operational amplifier) *operationsförstärkare*, angular(radian) frequency *vinkelfrekvens*,

# Solutions (EI1102, HT13, 2014-01-15)

**1)** Notes: the branch of $I$ $R_2$ behaves as a current source of value $I$, i.e. the resistor $R_2$ is irrelevant outside this branch; the combination of $U$ and $R_4$ behaves as a voltage source of value $U$, i.e. the resistor $R_4$ is irrelevant for other parts of the circuit.

**a)** Source $U$ fixes the voltage across $R_4$ (direct parallel connection). Thus, $i_4 = -U/R_4$.

**b)** Two convenient methods to find $u_1$ are nodal analysis and a source transformation.
Nodal Analysis: ground the bottom node; the node above $U$ forms part of a ground supernode; if we ignore the irrelevant $R_2$, then the only other node is the one above $R_1$. This node's potential is described by the marked (but unknown) voltage $u_1$. Then KCL gives $\frac{u_1}{R_1} + \frac{u_1-U}{R_3} - I = 0$. Thus $u_1 = \frac{(U+IR_3)R_1}{R_1+R_3}$.
Source Transformation: ignore the irrelevant $R_2$ and $R_4$; source-transform $U$ and $R_3$ into $R_3$ in parallel with a current-source $U/R_3$. The two current sources then have total current $I+U/R_3$, which returns through two parallel resistors $R_1$ and $R_3$. The voltage across $R_1$ is therefore $u_1 = (U/R_3 + I)\frac{R_1R_3}{R_1+R_3}$, as expected.
*Dimensional check:* The product $IR_3$ is [A][Ω] = [V] so it can correctly be added to $U$ [V]. The quotient $R_1/(R_1+R_3)$ is [Ω]/([Ω] + [Ω]) which is dimensionless. Thus the expression on the right-hand side has dimension [V], which matches with the left-hand side.

**2)** Let us define the centre node as ground, $v_0$. This is a good choice for simplifying the equations, as this node has the largest number of connections. Let us then define the other nodes $v_1$, $v_2$ and $v_3$, from left to right (the analysis shown here does not use the "supernode" method; but that could have been used as an alternative). Note that your solution might have defined the potentials differently. Define also a current $I_u$ into the + terminal of the voltage source $U$, and the marked voltage $u_x$ of the controlled current source $gu_x$. Summing outgoing currents, KCL at the three non-ground nodes gives:
KCL(1): $-I_u + \frac{v_1-v_2}{R_1} = 0$
KCL(2): $\frac{v_2-v_1}{R_1} + \frac{v_2-0}{R_2} + I = 0$
KCL(3): $-I - gu_x + \frac{v_3}{R_3} = 0$
Note that the above three equations contain three unknown potentials $v_1$, $v_2$ and $v_3$, *and* two more unknowns $I_u$ and $u_x$. Two further [linearly independent] equations are therefore needed:
Extra1: $u_x = v_1 - v_2$ (define controlling variable by node potentials)
Extra2: $v_1 = -U$ (voltage source relates node potentials)
The above set of 5 equations in 5 unknowns should be solvable. Direct use of the supernode method would have reduced this to two equations in two unknowns ($v_2$, $v_3$) but then we must include $v_1 = -U$, to define $v_1$.

**3)**

**a)** To find $u_c(0^-)$, redraw the circuit for equilibrium, i.e. for all time-derivatives being zero. Capacitors thus become open circuit, and inductors short circuit. The switch is still open-circuit at this time. From the redrawn diagram and a little application of the usual KCL/KVL, it is clear that $u_c(0^-) = U + IR_1$ .

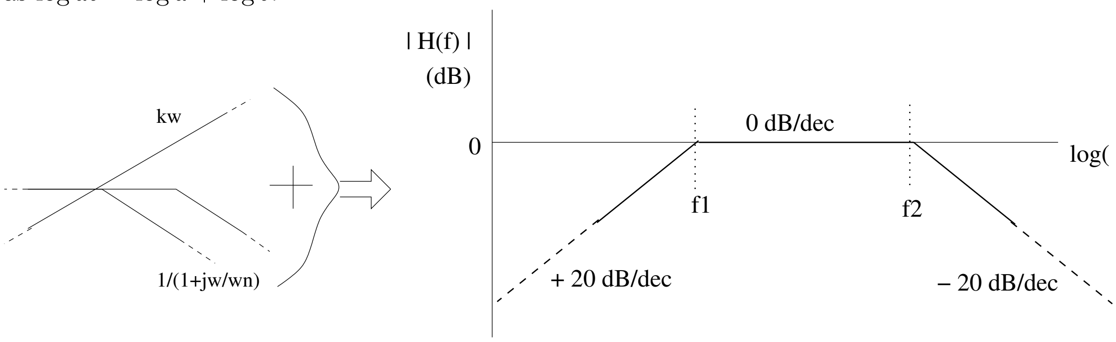
**b)** The voltage on a capacitor is continuous: it represents stored charge and energy, and cannot change instantaneously. Therefore, $u_c(0^+) = u_c(0^-) = U + IR_1$. When the switch is a short-circuit (like a zero voltage-source), the rest of the circuit (to the left) is irrelevant: the only components affecting $u_c(t)$ for $t \geq 0$ are $U$, $R_2$ and $C_2$. However, note that the initial voltage $u_c(0^+)$ *is* dependent on the history and therefore on the other components (from when the switch was not a short-circuit). Around the loop of the switch, voltage source, $C_2$ and $R_2$, we see that $\frac{U-u_c(t)}{R_2} = i(t)$ (from KVL around the loop, then Ohm's law), and $i(t) = C\frac{\mathrm{d}}{\mathrm{d}t}u_c(t)$ (from the relation of current and voltage in a capacitor), where $i$ is the current clockwise around this loop. Therefore, $\frac{\mathrm{d}}{\mathrm{d}t}u_c(t) + \frac{1}{C_2R_2}u_c(t) = \frac{1}{C_2R_2}U$. This ODE has solution $u_c(t) = U + A\mathrm{e}^{-\frac{t}{C_2R_2}}$ where $A$ is a constant to be determined from initial conditions. Knowing that $u_c(0) = U + IR_1$, we require that $U + IR_1 = U + A\mathrm{e}^0 = U + A$. Thus $A = IR_1$, so $u_c(t) = U + IR_1\mathrm{e}^{-\frac{t}{C_2R_2}}$.

**4)**

**a)** The first part of the circuit is not affected by the second; we see by voltage division that $u_1 = U \frac{1/(\mathrm{j}\omega C)}{R_1 + 1/(\mathrm{j}\omega C)}$, so $\frac{u_1}{U} = \frac{1}{1+\mathrm{j}\omega C R_1}$. In the second part, current division gives that $i_2 = g u_1 \frac{\mathrm{j}\omega L}{R_2+\mathrm{j}\omega L}$.
Putting these two parts together, $H(\omega) = \frac{i_2}{u_1}\frac{u_1}{U} = \frac{1}{1+\mathrm{j}\omega C R_1} \cdot \frac{g\mathrm{j}\omega L}{R_2+\mathrm{j}\omega L}$. You might have expressed this differently.

**b)** In the solution above, divide the top and bottom of the right-hand term by $R_2$. This brings it to the desired form $H(\omega) = \frac{k\,\mathrm{j}\omega}{(1+\mathrm{j}\omega/\omega_1)(1+\mathrm{j}\omega/\omega_2)}$, where $k = gL/R_2$ and $\omega_1$ and $\omega_2$ are $\frac{1}{R_1 C}$ and $\frac{R_2}{L}$ (in either order).

**c)** The three terms containing $\omega$ can be sketched separately, as shown on the left in the diagram. In a log/log (or dB/log) plot, the asymptotic approximation allows straight lines to be used. The log-scaling provided by dB values also allows the magnitudes of the separate terms to be *added* or *subtracted* to find the total function, as $\log ab = \log a + \log b$.

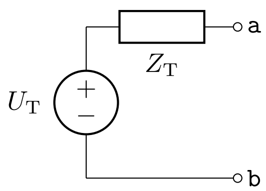



Between $\omega_1$ and $\omega_2$ the amplitude of this asymptotic Bode amplitude-plot is 0 dB. As we move towards lower or higher frequencies outside this band, the amplitude falls off att 20 dB/decade. This is the behaviour of a band-pass filter. The fact that $k = 1/\omega_1$ tells us that the amplitude of the term $\mathrm{j}\omega k$ is 1 (0 dB) at $\omega_1$.

**5)**

**a)** The opamp output voltage is not affected by $R_\mathrm{o}$ and $L$ (its output voltage must be whatever is needed to force the inverting and non-inverting inputs to be at the same potential). At the inverting input, the potential must be zero ("virtual ground" to match the grounded non-inverting input), so no current flows in $R_\mathrm{i}$. The feedback current $(v_\mathrm{o} - 0)\mathrm{j}\omega C$ must therefore equal the current-source $I$, so $v_\mathrm{o} = \frac{I}{\mathrm{j}\omega C}$. This could be found more formally by nodal analysis.

**b)** The open circuit voltage (marked $u_\mathrm{ab}$) is the same thing as the Thevenin voltage.
By voltage division of $v_\mathrm{o}$, this is $U_\mathrm{T} = u_\mathrm{ab} = \frac{\mathrm{j}\omega L}{R_\mathrm{o}+\mathrm{j}\omega L} v_\mathrm{o} = \frac{\mathrm{j}\omega L}{R_\mathrm{o}+\mathrm{j}\omega L} \cdot \frac{1}{\mathrm{j}\omega C} \cdot I$, so $U_\mathrm{T} = \frac{\mathrm{j}\omega L}{\mathrm{j}\omega C(R_\mathrm{o}+\mathrm{j}\omega L)} I$.
The Thevenin impedance can be found via calculating the short-circuit current (e.g. short-circuit `a` and `b`, and find the current between them, which is the same as through $R_\mathrm{o}$). Or it can be found more directly by noticing that $v_\mathrm{o}$ behaves like an ideal voltage source, and thus the Thevenin impedance is the parallel combination of $R_\mathrm{o}$ and $L$, i.e. $Z_\mathrm{T} = \frac{\mathrm{j}\omega L R_\mathrm{o}}{R_\mathrm{o}+\mathrm{j}\omega L}$. Rearranging in a convenient form for part 'c)', $Z_\mathrm{T} = \frac{\omega L R_\mathrm{o}}{R_\mathrm{o}^2+\omega^2 L^2}(\omega L + \mathrm{j}R_\mathrm{o})$.




**c)** Maximum power into a load impedance $Z$ from a Thevenin source with impedance $Z_\mathrm{T}$ requires that $Z = Z_\mathrm{T}^*$. Therefore, based on the solution of part 'b)', $Z = \frac{\omega L R_\mathrm{o}}{R_\mathrm{o}^2+\omega^2 L^2}(\omega L - \mathrm{j}R_\mathrm{o})$.

**6)**

**a)** Using a cosine reference and peak-values, the voltage source in the frequency-domain (j$\omega$) has the phasor $u(\omega) = \hat{U}\angle\varphi$; this question did not specify the reference, so other choices could have been made (the final time-domain solution should be identical in any case). The total impedance in the circuit is due to the series inductor and capacitor, $Z = R + \mathrm{j}\omega L$. Therefore, the current is $i(\omega) = \frac{u(\omega)}{Z} = \hat{U}\angle\varphi \,/\, (R + \mathrm{j}\omega L)$.
We could multiply top and bottom (numerator and denominator) by the complex conjugate of the bottom, to force the bottom to be real; but all we really want is the amplitude and phase of this complex number, and these are easily found using polar form and the usual relation that $A\angle\alpha / B\angle\beta = \frac{A}{B}\angle\alpha - \beta$. Thus, $|i(\omega)| = \frac{\hat{U}}{\sqrt{R^2+\omega^2L^2}}$ and $\angle i(\omega) = \varphi - \tan^{-1}(\omega L/R)$. The required solution is the time-domain current. Remembering that we have used a cosine reference for angles, and peak-values for amplitudes, we can write $i(t) = |i(\omega)| \cos(\omega t + \angle i(\omega))$. Therefore, $i(t) = \frac{\hat{U}}{\sqrt{R^2+\omega^2L^2}} \cos\left(\omega t + \varphi - \tan^{-1}\frac{\omega L}{R}\right)$.

**b)** Complex power is $S = P + \mathrm{j}Q = UI^* = |U||I|\angle(\angle U - \angle I)$, where the amplitudes of phasors $U$ and $I$ are rms (if they are peak values, divide the result by 2 to give $S$). Calculation of $S$ in a circuit element therefore requires that we know the amplitudes and *relative* phases of voltage and current for that element.
For the voltage source, we know from part 'a)' the source's voltage and current in the time-domain and frequency-domain. We see that they are defined in '*active* convention' and are *peak* values.
Therefore, the complex power *out* of the source is $S = \frac{1}{2}u(\omega)i(\omega)^*$, which is $\frac{1}{2}\frac{\hat{U}^2}{\sqrt{R^2+\omega^2L^2}}\angle\tan^{-1}\frac{\omega L}{R}$. In the case of complex power, it is usually more helpful to use rectangular form to show $P$ and $Q$ directly: hence, $S = \frac{1}{2}\frac{\hat{U}^2}{R^2+\omega^2L^2}(R + \mathrm{j}\omega L)$. But any correct form of the answer is acceptable, if suitably simplified.

**c)** The final tricky part! Remember that the voltage in a coupled coil depends on the current in this coil *and* the other[s]. The voltage in the second (open-circuit) coil is $u_2(t) = \pm L_2\frac{\mathrm{d}}{\mathrm{d}t}i_2(t) \pm M\frac{\mathrm{d}}{\mathrm{d}t}i_1(t)$, where the $\pm$ depends on how the currents and voltages are defined (dot-convention between the coils, passive/active convention for $u$ and $i$). In our case, we know $i_2(t) = 0$ (open-circuit) and we do not care about the *sign* of $M\frac{\mathrm{d}}{\mathrm{d}t}i_1(t)$ as we only have to find the amplitude.
The frequency-domain relation for the first coil is simpified by omitting the mutual-inductance term $M\frac{\mathrm{d}}{\mathrm{d}t}i_2(t)$, which we know must be zero as $i_2 = 0$. Using passive convention for $u_1$ and $i_1$, the simplified relation is then $u_1(\omega) = \mathrm{j}\omega L_1 i_1(\omega)$. This means the the pair of coupled coils behaves like an inductor of value $L_1$ when seen from the rest of the circuit. The magnitude of current $i_1(\omega)$ can therefore be expressed by replacing $L$ with $L_1$ in the amplitude expression that we found in part 'a)'.
The magnitude of $i_1$ can then be used to find the magnitude of voltage $u_2$, which is $|u_2(\omega)| = \omega M|i_1(\omega)|$ due to our knowledge that $i_2 = 0$. Naturally, the question didn't simply provide $M$, as tradition dictates that questions provide all the values except the important one… you have to use the relation $M = k\sqrt{L_1L_2}$.
The magnitude of voltage on the open coil is then $|u_2(\omega)| = \frac{\hat{U}\omega k\sqrt{L_1L_2}}{\sqrt{R^2+\omega^2L_1^2}}$.

# EI1102 / EI1100 Exam, VT13: 2014-01-15.

## Questions asked during the exam

Sorted by approximate descending popularity. Several were from just one person.

Q3a. *Can we assume steady-state (jämviktsläge)?*
(A) Yes. The clue "constant over all time" and the implication that the only change in all time is the switch at $t = 0$, tells us that at $t = 0^-$ the circuit has been standing "since $t = -\infty$", so we expect a steady state. Next time I think I will include "anta att kretsen är i jämviktsläge vid tid $t = 0^-$" anyway, to reduce the number of questions!

Q5. *Should $v_o$ be given in frequency-domain or time-domain?*
Frequency-domain. It was admittedly not specified. Note, however, that the normal meaning of a Thevenin equivalent of a circuit with $C$ or $L$ is only valid as a frequency-domain view. It could be better to have an explicit argument $v_o(t)$ or $v_o(\omega)$ in every case, but it becomes rather unsightly when every quantity in an equation carries these arguments.

Q4. *What about $U$: is this* not *a known quantity?*
(A) It's not a circuit-constant. In general we expect it to be a complex number that is a function of $\omega$. You can regard it as known if you like: it is the only driving force (input source) to the [linear] circuit, and it therefore comes into all expressions for current or voltage, as a multiplied term. For this reason, it disappears (cancels) in the final expression for $H(\omega)$ ... so you do not need to assume it has any particular value.

Q3b. *You said there would not be "find $u(t), t \geq 0$" types of question for more than one [independent] $C$ or $L$ component". Yet here we see $L_2$, $L_1$, $C_1$, $C_2$... is this your idea of a joke, giving us much more complex problems than you had promised? In that case I'm afraid I fail to see the humour...*
(A) See the second line about how the part to the left of the switch is not needed for the differential-equation solution. Consider the "irrelevance" principle. The closed switch (starting at $t = 0$) is a short-circuit, which is like a zeroed voltage-source. This, in parallel between two nodes, forces them to the same potential. There is no way for components on its left to influence the circuit during the period that the differential equation has to describe. So the solution is just for the $U$, $C_2$, $R_2$ components.

Q4c. [**My favourite question.**] *You say "assume $k = 1/\omega_1$", but I've just done a full dimensional check of my working, and seen that $\omega_1$ has dimension $[\mathrm{s}^{-1}]$, while $k$ needs to have dimension $[\mathrm{s}][\Omega^1]$. So this can't be right.*
(A) Congratulations! I confess had not even thought about this. The intention was that this was a way of saying "assume that $g$ has a magnitude of 1"; a more precise way to define $g$ would be $g = 1\,\mathrm{S}$ (S is siemens, which means $\Omega^{-1}$). So I meant the numerical values of $1/\omega_1$ and $k$ are the same. More properly I could have said $\frac{k}{[\mathrm{s}\cdot\mathrm{S}]} = 1/\frac{\omega_1}{[\mathrm{s}^{-1}]}$. I think that would have confused more people! But I will certainly think harder about this for other times. Very well done for such a detailed analysis.

Q4a. *Could I lose marks for not simplifying the answer?*
(A) Good question, bearing in mind that part 'b' shows a very simplified version: so, what if one simplifies in part 'b'? Does one get the mark in part 'b', but lose marks in part 'a' for not having simplified there? Clearly one should simplify very obvious things like $b(a - b - a)/b$ in answers. However, for this question we shall say that you should get full marks if the answer in part-a is *correct*, *and* part-b correctly shows it to match the given function; it will not matter in which part the simplifications are done.

Q2. *Won't this have different numbers of equations, depending on which node is ground?*
(A) Many different answers are possible, depending on how the intermediate variables like $u_x$ are handled, and whether the supernode method is used. Basically, it's a matter of whether you simplify the equations before writing them up (e.g. substitute $u_x = v_1 - v_2$), or whether you keep these facts as separate equations. But

ultimately it should come down to the right number to determine all the 3 node-potentials with respect to the 4th node, which has been defined as ground (0). We will of course be very tolerant of whether answers include a statement $u_{\text{ground}} = 0$ since this should also shown in your diagram. However, a supernode (like the one on the left) should have its potential written: it is not enough just to give two equations for the two other nodes and not even mention that e.g. $v_1 = -U$. (The examples above are based on the centre being the ground: you may have made another choice.)

Q6. *How do we know is $\hat{U}$ is peak or rms? We need to know whether to have the factor $\frac{1}{2}$ in the expression for power.*
The given format is time-domain, from which you can see whether the voltage has peak or rms $\hat{U}$. It's only if we give a frequency-domain value (a phasor) of voltage or current, that you don't know our choice of how this relates to the time-domain signal. There are two things: amplitude (peak, rms, something-else?), and phase (cosine- or sine-reference, or something else?). In some cases it doesn't matter: you can solve for other voltages and currents without knowing, and they will be in the same reference system. For power, or for converting back to time-domain, these details can, however, matter. Note that "real users" of ac analysis very often do everything in phasors. In power subjects, at least, these are almost always rms.

Q1. *Can one choose any reference direction one wants for one's own definitions (intermediate steps)?*
(A) Yes. As long as you get a correct result, you choose your own intermediate definitions.