Tentamen [s]: EI1102/EI1100 Elkretsanalys, 2014-08-29 kl 08–13

Omtentan gäller en nerlagd kurs, vilken hade sista omgången HT13. Nästa omtentan planeras att ske samtidig med tentan i kursen EI1120, VT15.

Hjälpmedel: *Ett* A4-ark med studentens anteckningar (båda sidor). Dessutom, pennor!

Svar får anges på svenska eller engelska. En kort ordlista finns på sista sidan.
Tentan har 3 tal i del A (15p), och 3 tal i del B (15p).
Godkänt vid $\geq$25% på del A och del B individuellt, *och* $\geq$50% på delarna A och B tillsammans. Betyget räknas sedan från summan av A och B. Se KursPM:et angående betygsgränser, rättningsnormer och överklagande.

Läs varje tal noggrant **innan du försöker svara**.

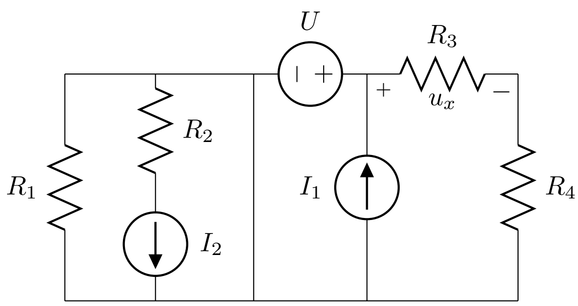
Senare *del*tal kan vara betydligt **svårare** än de första deltalen.
Satsa inte för mycket tid på bara en uppgift om du fastnar: ta hänsyn till poängvärden på uppgifterna, och att man måste både delar av tentan.

Tänk på att **använda återstående tid till att kolla igenom varje svar**: man kan göra dimensionsanalys, rimlighetsbedömning (t.ex. ”är det rätt att y går ner medan x går ner?”), och lösning genom en alternativ metod. Lösningar ska **förenklas** om inte annat är specificerat.

Examinator: Nathaniel Taylor

Del A. Likström och Transienter.

1) [5p]
Kända: R_1, R_2, R_3, R_4, U, I_1, I_2.



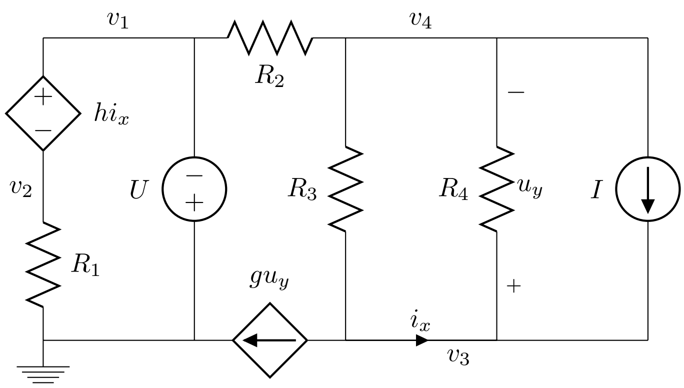
a) [1p] Bestäm effekten levererat till R_2.

b) [1p] Bestäm den markerade spänningen u_x.

c) [1p] Bestäm effekten levererat till R_3.

d) [2p] Bestäm effekten levererat från källan U.

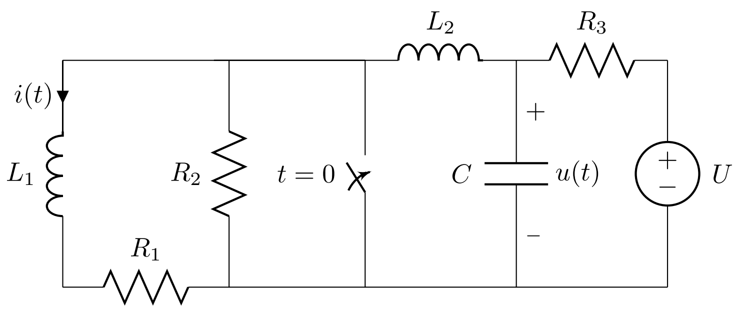
2) [5p]
Kända: R_1, R_2, R_3, R_4, U, I, g, h.



Använd nodanalys för att skriva ekvationer som går att lösa för de okända potentialerna v_1, v_2, v_3, v_4.

Du *måste inte lösa* ekvationerna, och *måste inte* skriva om dem i förenklad eller matris form. Det finns flera möjliga svar (alla med samma lösning).

3) [5p]
Kända: U, C, L_1, L_2, R_1, R_2, R_3.
Kretsen är i jämviktsläge vid tiden $t = 0^-$, d.v.s. just innan $t = 0$ när brytaren stängs.



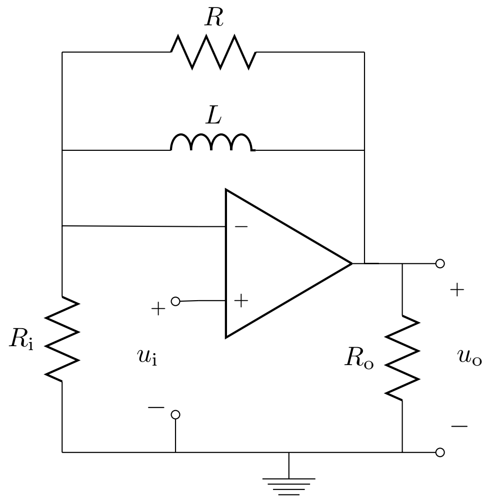
a) [2p] Bestäm $i(0^-)$.

b) [2p] Bestäm $u(0^+)$.

c) [1p] Bestäm funktionen $i(t)$, för $t > 0$.

Del B. Växelström

4) [5p]
Kända: R_i, R, L, R_o.
Spänningen u_i på polerna till vänster orsaker spänningen u_o mellan polerna till höger.



a) [2p] Bestäm kretsens nätverksfunktion, $\dfrac{u_\mathrm{o}(\omega)}{u_\mathrm{i}(\omega)}$.

b) [3p] Nätverksfunktionen från deltal 'a)' borde kunna skrivas i formen,

$$H(\omega) = 1 + \frac{\mathrm{j}\omega/\omega_\mathrm{z}}{1 + \mathrm{j}\omega/\omega_\mathrm{p}}.$$

(Du måste inte bevisa det, men du kanske vill dubbelkolla del 'a'!) Skissa ett Bode amplituddiagram av $H(\omega)$, med viktiga punkter och lutningar markerade. Anta att $\omega_\mathrm{p} \gg \omega_\mathrm{z}$. Är du osäker om hantering av '1+' termen, så kan du lämna ut den utan stort avdrag.

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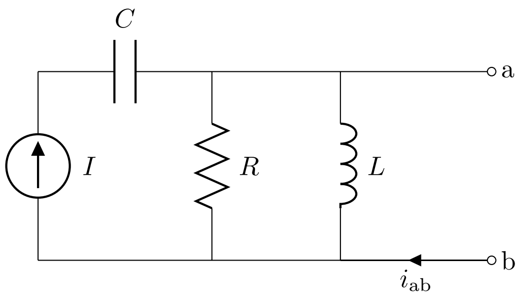
5) [5p]
Kända: I, C, R, L, ω.
I är en komplex storhet som beskriver en växelströmskälla av vinkelfrekvensen ω.



a) [2p] Bestäm Norton-ekvivalenten av kretsen, mellan polerna 'a' och 'b'. Visa ekvivalanten med ett diagram. Var inte förvånad om inte alla kända komponentvärdena behövs.

b) [3p] En impedans kopplas till kretsen, mellan polerna 'a' och 'b'. Rita ett diagram av hur impedansen kan skapas med två komponenter (välj mellan motstånd, spole, kondensator) på ett sätt som maximerar överföringen av aktiveffekt till impedansen. Det finns flera än ett sätt. Uttryck de två komponentvärden i de kända variablerna.

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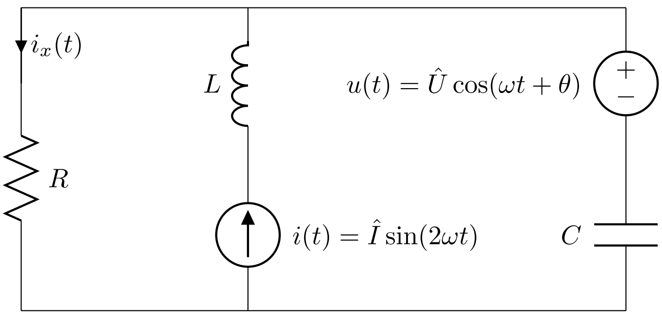
6) [5p]
Kända: $\hat{U}$, $\hat{I}$, ω, θ, R, L, C.
Tidsvariabeln är t.



a) [3p] Bestäm $i_x(t)$. Anta att lösningen kan göras med växelströmsanalys, d.v.s. att det inte finns transienter kvar. Observera att källorna har olika frekvenser.

b) [2p] Bestäm den genomsnittliga effekten ("aktiveffekt" i växelströms terminologi) i R.

Ordlista över mindre självklara översättningar: resistor *motstånd*, capacitor *kondensator*, inductor (coil) *spole*, current *ström*, voltage *spänning*, power *effekt*, active power *aktiveffekt* source *källa*, switch *brytare*, terminal *pol*, angular(radian) frequency *vinkelfrekvens*,

Solutions (EI1102, Omtenta 2014-08-29)

Q1

a) Resistor R_2 is in series with a current source I_2, which determines its current. Power in a resistor can be calculated from current and resistance, without having to consider the direction of current. The power into the resistor is $I_2^2 R_2$.

b) The marked voltage u_x across R_3 can be found by voltage division across the series-connected pair R_3 and R_4. This is convenient because this pair is directly connected to the voltage source U; it might not be very obvious, but you can follow the nodes round to see that it's true. Therefore, $u_x = U\frac{R_3}{R_3+R_4}$. One should be careful to check that the definition directions of U and u_x don't make a negative sign necessary!

c) The voltage across R_3 is u_x, already known from the above.
The power dissipated in a resistor can be found from the voltage across it and its resistance, i.e. $\frac{u_x^2}{R_3}$.
This power is $\left(U\frac{R_3}{R_3+R_4}\right)^2/R_3$, which simplifies to $\frac{U^2 R_3}{(R_3+R_4)^2}$.

d) The voltage of a voltage source is known by definition. By finding the current coming out of the + terminal of source U, and multiplying by this voltage U, the power out of the source is found. So, what *is* the current? Consider the node at the + terminal of the voltage source:
* the current from this node into the current source I_1 is $-I_1$;
* the current into the resistor R_3 is $\frac{U}{R_3+R_4}$ as found in part 'b)';
and so, by KCL, the current out from the voltage source's + terminal is $\frac{U}{R_3+R_4} - I_1$.
The power out from the voltage source is then $\frac{U^2}{R_3+R_4} - UI_1$.

Q2

Some previous solutions of homeworks or exams have shown two methods of systematically writing an equation system from a circuit. Here, we will use just the 'simple' direct method, which results in more equations and variables, but has advantages of less work spent on making simplifications, and more direct correspondence between the equations and the circuit diagram. Other methods, e.g. by considering 'supernodes', produce a smaller equation system, which can be desirable for hand-calculation.

Define currents (unknown variables) in the voltage sources: let's call them I_α into the + pole of source U, and I_β into the + pole of source hi_x.

Now do KCL on the outgoing current at all the nodes other than the ground node:

KCL(1): $I_\beta - I_\alpha + \dfrac{v_1 - v_4}{R_2} = 0$

KCL(2): $-I_\beta + \dfrac{v_2}{R_1} = 0$

KCL(3): $gu_y + \dfrac{v_3 - v_4}{R_3} + \dfrac{v_3 - v_4}{R_4} - I = 0$

KCL(4): $\dfrac{v_4 - v_1}{R_2} + \dfrac{v_4 - v_3}{R_3} + \dfrac{v_4 - v_3}{R_4} + I = 0$

Then we add equations describing the relations imposed between node potentials by the voltage sources. These two equations compensate for the two unknown currents I_α and I_β in the voltage sources.

$v_1 = -U$

$v_1 - v_2 = hi_x$

Finally, the two controlling variables of the dependent sources need to be defined in terms of already-introduced variables:

$u_y = v_3 - v_4$

$$i_x = \frac{v_3 - v_4}{R_4} - I$$

Counting up the unknowns and equations, we see $v_{1\cdots4}$, I_α, I_β, i_x and u_y (8 unknowns), and 8 equations also. Having used a suitable systematic method, we can be confident that a solvable circuit will have given a solvable equation system.

Q3

a) The source is dc (a constant value) and we are told to assume that there is equilibrium at this time $t = 0^-$. We can therefore find voltages and currents by replacing inductors with short circuits, and capacitors with open circuits. This leaves a circuit with just the three resistors and voltage source.

Solving for current i, which is the current through resistor R_1, we can first find the current from the source, which is $\frac{U}{\frac{R_1R_2}{R_1+R_2}+R_3}$, then use current division between R_1 and R_2, giving $i(0^-) = \frac{U}{\frac{R_1R_2}{R_1+R_2}+R_3} \cdot \frac{R_2}{R_1+R_2}$.

Simplifying this, $i(0^-) = \frac{UR_2}{R_1R_2+R_3(R_1+R_2)}$.

b) The voltage $u(t)$ is defined across a capacitor: it is therefore a *continuous* variable, as it relates to the stored energy, so $u(0^-) = u(0^+)$. The task of finding $u(0^+)$ is thus acheived by finding the equilibrium level, $u(0^-)$ *before* the switch closed.

By the same process as part 'a)', the circuit can be reduced to three resistors and the voltage source, and u is seen to be the same as the voltage across the parallel resistors R_1 or R_2.
By voltage division, this is $u(0^-) = U\frac{\frac{R_1R_2}{R_1+R_2}}{\frac{R_1R_2}{R_1+R_2}+R_3}$.
Simplifying, and using $u(0^-) = u(0^+)$, the answer is $u(0^+) = U\frac{1}{1+R_3\frac{R_1+R_2}{R_1R_2}}$ or $u(0^+) = \frac{UR_1R_2}{R_1R_2+R_2R_3+R_3R_1}$.

c) The current i is the current in an inductor, so it is a continuous variable. Its initial state, $i(0^+)$, is therefore the same as $i(0^-)$ which was found in part 'a)'. The switch short-circuits R_2 and the right part of the circuit, so the current $i(t > 0)$ runs in a loop of just L_1 and R_1. This loop has a time-constant of $\tau = L_1/R_1$. The final state of $i(t)$ as $t \to \infty$, must be zero, as there is no source but there is a resistor that dissipates energy.

The requested time-function is therefore $i(t) = i(0^-)\exp(-t/\tau)$, which has the desired property of going from the initial value to the final value, in an exponential shape with time-constant τ, as expected for a first-order circuit.

This answer should ideally be expressed in the given quantities: $i(t) = \frac{UR_2}{R_1R_2+R_3(R_1+R_2)} \exp -t\frac{R_1}{L_1} \quad (t > 0)$.

Q4

Yes, I could have mentioned that the opamp is ideal. This time I thought I wouldn't, as this is an assumption we make for all the components. But previous exams always said this, so I understand it was a bit worrying if something was different in this case.

a) The function $u_\mathrm{o}/u_\mathrm{i}$ can be found by assuming that the opamp's feedback is forcing the inverting input to have potential u_i (to follow the non-inverting input). Then voltage division of u_o between the two impedances provides the necessary relation between u_o and u_i: $u_\mathrm{i} = u_\mathrm{o}\frac{R_i}{R_i+\frac{R\mathrm{j}\omega L}{R+\mathrm{j}\omega L}}$.

The desired network function requires this to be inverted to give the ratio $u_\mathrm{o}/u_\mathrm{i}$: $\frac{u_\mathrm{o}}{u_\mathrm{i}} = 1 + \frac{\mathrm{j}\omega L/R_i}{1+\mathrm{j}\omega L/R}$.
This could be written in different forms, e.g. by changing terms like R/L for a frequency ω_x, or by keeping terms $(R+\mathrm{j}\omega L)$ instead of making them dimensionless; the question did not state that the canonical form is required.

b)

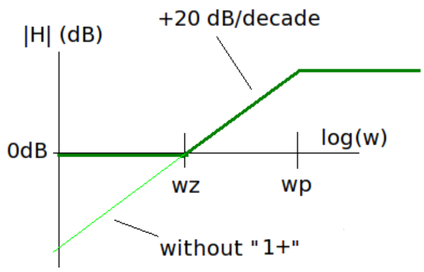
Ignoring the '1+' part of $H(\omega)$, there is a +20 dB/decade slope for $\omega < \omega_\mathrm{p}$, and a 0 dB/decade slope for $\omega > \omega_\mathrm{p}$. In the region $\omega = \omega_\mathrm{z}$, the amplitude passes through 0 dB.

This shape can be understood by combining the separate effects of the numerator, $\mathrm{j}\omega/\omega_\mathrm{z}$, which has +20 dB/decade slope everywhere, and the denominator, $1 + \mathrm{j}\omega/\omega_\mathrm{p}$, which has −20 dB/decade slope at $\omega > \omega_\mathrm{p}$ and flat 0 dB at $\omega < \omega_\mathrm{p}$).

To include the '1+' term, notice that this ensures that $|H(\omega)|$ does not fall below 1 (0 dB). (This claim depends on the main term not having a negative real part; in our case this is true, as the main term's phase varies only from 90° to 0°.

In the given situation of $\omega_z \ll \omega_p$, all the region $\omega > \omega_z$ would have $|H(\omega)| > 1$, so the '1+' is negligible. The Bode plot can be modified to include the '1+' by forcing the plot up to a constant 0 dB for $\omega < \omega_z$.

One could also mention that the gain at the high frequencies, $\omega > \omega_p$, is approximately $20 \log_{10} \frac{\omega_p}{\omega_z}$. Pedantically, we could say the points on the horizontal axis should be marked as e.g. $\log \omega_p$ instead of ω_p.



Q5

a) Component C is irrelevant to the rest of the circuit, as it is in series with a current source. As we are only interested in what happens outside the branch of I and C, we can replace these two by just I.

The circuit is then almost exactly a Norton equivalent already.
The Norton current source is $I_N = I$, still in the same direction with regard to terminals 'a' and 'b'.
The conductance $\frac{1}{R}$, and susceptance $\frac{1}{\omega L}$, are combined into a single admittance, $Y_N = \frac{1}{R} + \frac{1}{j\omega L}$.
As an alternative (less conventional for Norton sources) we can write this as an impedance, $Z_N = \frac{j\omega LR}{R+j\omega L}$.

The same result could be found by the more general way, of calculating short-circuit current and open-circuit voltage.

b) The ac maximum power-transfer condition is that the load's impedance (or admittance) equals the complex conjugate of the impedance (or admittance, respectively) of the source.

So we want to make a load admittance Y_x such that $Y_x = Y_N^* = \frac{1}{R} - \frac{1}{j\omega L}$, or equivalently, $Y_x = \frac{1}{R} + j\frac{1}{\omega L}$.

Notice that this admittance has a positive real part and positive imaginary part. A resistor has purely positive real impedance or admittance. An inductor or capacitor has a purely imaginary impedance or admittance, which can be negative or positive depending on the component and on whether we use impedance or admittance.

When connecting admittances in parallel, they sum directly to the total admittance (impedances sum when series). The positive real and imaginary parts of the required admittance can therefore be independently implemented with a resistor R_x and capacitor C_x. These give an admittance of $\frac{1}{R_x} + j\omega C_x$. We have to choose R_x and C_x to give the required value of Y_x.

Considering the real and imaginary parts separately, we see $R_x = R$, and $j\omega C_x = j\frac{1}{\omega L}$, i.e. $C_x = \frac{1}{\omega^2 L}$.

The other way to implement the right load for maximum power would be a series connection of a resistor and capacitor. In this case, the necessary values would be different from the ones needed for a parallel connection: let's call them R_y and C_y. The criterion would be $R_y - j\frac{1}{\omega C_y} = Z_y = Z_N^* = \frac{1}{Y_N^*}$. It is useful to get this last term into a form where the real and imaginary parts are separate so that they can be equated with R_y and $\frac{1}{\omega C_y}$.
Hence $\frac{1}{Y_N^*} = \frac{1}{\frac{1}{R} - \frac{1}{j\omega L}} = \frac{j\omega LR}{j\omega L - R} = \frac{\omega^2 L^2 R - j\omega L R^2}{R^2 + \omega^2 L^2} = R_y - j\frac{1}{\omega C_y}$.

From this, $R_y = \frac{\omega^2 L^2 R}{R^2 + \omega^2 L^2}$, and $-j\frac{1}{\omega C_y} = -j\frac{\omega L R^2}{R^2 + \omega^2 L^2}$, i.e. $C_y = \frac{R^2 + \omega^2 L^2}{\omega^2 L R^2}$.

Notice that the choice of parallel components was simpler in this case, as it better matches the known components R and L in the source circuit.

Q6

a) The independent sources have different frequencies. (It might have been clearer just to have written them as ω_1 and ω_2, instead of ω and 2ω.) Superposition is therefore *needed* if we are going to take advantage of the convenient method of steady-state sinusoidal analysis, which solves for a specific frequency.

Consider the contribution of each source separately, to the marked current $i_x(t)$.

Case 1. With just the current source active, the voltage source is a short-circuit. The circuit can be redrawn with the irrelevant inductor omitted and the voltage source being a short-circuit; then we see that the current at frequency 2ω divides through R and C. Let's define the current source in the frequency-domain as being a current phasor $I = \hat{I}\angle 0$, which means we've used a sine convention and peak-value convention.

Then by current-division, $i_{x(1)}(2\omega) = I\frac{\frac{1}{\mathrm{j}2\omega C}}{R+\frac{1}{\mathrm{j}2\omega C}} = \frac{1}{1+\mathrm{j}2\omega CR}I = \frac{\hat{I}}{\sqrt{1+4\omega^2C^2R^2}}\angle -\tan^{-1}\frac{2\omega CR}{1}$.

Now returning to the time-domain, remembering to use the same sine- and peak conventions as we used at the start, $i_{x(1)}(t) = \frac{\hat{I}}{\sqrt{1+4\omega^2C^2R^2}}\sin\left(2\omega t - \tan^{-1}(2\omega CR)\right)$.

Case 2. Now with just the voltage source active, the current source can be ignored (open circuit), and the circuit is simply a loop of U, R, C. Let us define the voltage source as $U = \hat{U}\angle\theta$, which means we've used a cosine reference and peak values. Notice that the reference can be independently chosen for each superposition-case; what *is* important is to choose the same reference for the time-frequency conversion as for the frequency-time conversion, for any particular case, in order to get the correct time-functions.

The marked current in this second case is $i_{x(2)}(\omega) = \frac{U}{R-\frac{\mathrm{j}}{\omega C}} = \frac{\hat{U}}{\sqrt{R^2+\frac{1}{\omega^2C^2}}}\angle\theta + \tan^{-1}\frac{1}{\omega CR}$.

In the time-domain, using the cosine reference that we used when converting to the frequency-domain, this current is $i_{x(2)}(t) = \frac{\hat{U}}{\sqrt{R^2+\frac{1}{\omega^2C^2}}}\cos\left(\omega t + \theta + \tan^{-1}\frac{1}{\omega CR}\right)$.

Finally: By superposition, the actual current $i_x(t)$ is the sum of both the above results,

$$\boxed{i_x(t) = \frac{\hat{I}}{\sqrt{1+4\omega^2C^2R^2}}\sin\left(2\omega t - \tan^{-1}2\omega CR\right) + \frac{\hat{U}}{\sqrt{R^2+\frac{1}{\omega^2C^2}}}\cos\left(\omega t + \theta + \tan^{-1}\frac{1}{\omega CR}\right)}.$$

b) The mean power (over multiple periods of the source frequencies) could be found by integrating instantaneous power over some time T during which a whole number of cycles of both frequencies have occurred, i.e. $\bar{P} = \frac{1}{T}\int_t^{t+T} Ri^2(t)\mathrm{d}t$.

But there is a more convenient method, power superposition, which is valid only for this situation where the sources being handled by superposition have different frequencies.

Looking back to the frequency-domain expressions for $i_x(\cdot)(\omega)$ in the working of part 'a)', the active power in each superposition case is easily found by squaring the *magnitude* of the current, multiplying by the resistance, and dividing by 2 (because of peak values being used).

Summing the results for the two frequencies, this is $\boxed{\frac{R}{2}\left(\frac{\hat{I}^2}{1+4\omega^2C^2R^2} + \frac{\hat{U}^2}{R^2+\frac{1}{\omega^2C^2}}\right)}$.