KTH EI1102 / EI1100 Elkretsanalys, Tentamen 2015-03-17 kl 08–13

Tentan har 6 tal i 2 delar: tre tal i del A (15p), och tre i del B (15p).

Hjälpmedel: Ett A4-ark (båda sidor) med studentens egna anteckningar på valfritt sätt: handskrivet eller datorutskrift; text eller diagram; stor eller liten textstorlek, . . .).

Om inte annan information anges i ett tal, ska: komponenter antas vara idéala; angivna värden av komponenter (t.ex. R för ett motstånd, U för en spänningskälla, k för en beroende källa) antas vara *kända* storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller en spänningskälla) antas vara *okända* storheter. Lösningar ska uttryckas i kända storheter, och förenklas. Var tydlig med diagram och definitioner av variabler.

Tips: Dela tiden mellan talen. Senare deltal brukar vara svårare att tjäna poäng på: fastna inte på dessa. Kontrollera svarens rimlighet genom t.ex. dimensionskoll eller alternativ lösningsmetod. Det kan hjälpa att rita om ett diagram för varje tillstånd, t.ex. innan och efter en brytare ändrar kretsen, eller med ersättningar eller borttagning av delar som inte är relevant för det sökta värdet. Då blir kretsen ofta mycket lättare att tänka på och lösa.

För godkänd (E och högre) krävs minst 25% i del A, och minst 25% i del B, samt minst 50% räknat över båda delarna. Betyget räknas då över båda delarna: gränserna (%) är 50 (E), 60 (D), 70 (C), 80 (B), 90 (A). Om tentan blev underkänd med liten marginal, så kan betyget Fx registreras, med möjlighet att få betyget E om ett kompletteringsarbete är godkänt inom några veckor efter tentamen.

Examinator: Nathaniel Taylor

Del A. Likström och Transient

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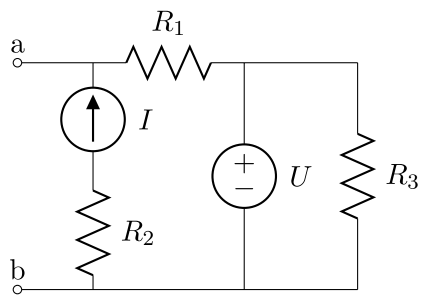
1) [5p]

a) [4p] Bestäm Theveninekvivalenten av kretsen, med avseende på polerna 'a' och 'b'. Rit upp ekvivalenten inklusive polerna.

b) [1p] Vilken maximaleffekt kan levereras från kretsen till en last kopplad mellan polerna a-b?



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2) [5p]

Bestäm de följande:

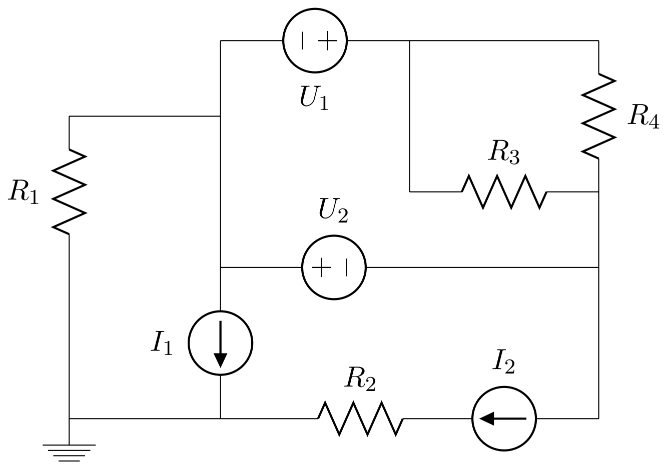
a) [1p] Effekten levererat till R_2.

b) [1p] Effekten levererat till R_3.

c) [1p] Effekten levererat till R_1.

d) [1p] Effekten levererat från källan U_1.

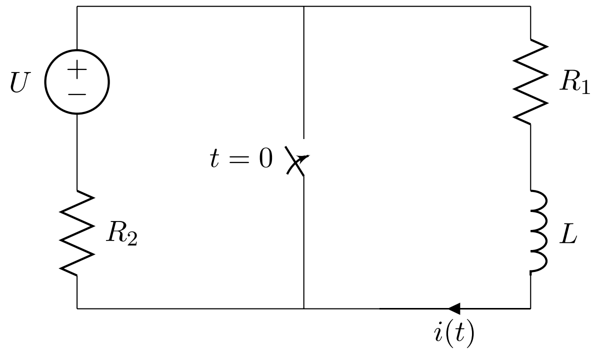
e) [1p] Effekten levererat från källan I_2.



3) [5p]

Bestäm $i(t)$, för $t > 0$.

(Ja, jämviktsläge antas innan. Det borde inte behöva specifieras, då det står inte att någon annan ändring har skett mellan 'länge sedan' och $t = 0$.)

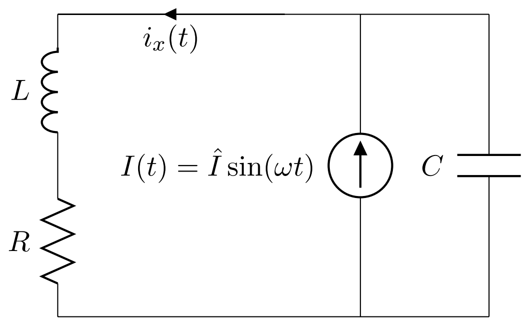


Del B. Växelström

4) [5p]

a) [4p] Bestäm $i_x(t)$. Växelströmsjämviktsläge gäller: d.v.s. 'jω-metoden' kan användas. Det får antas att $\frac{1}{\omega C} > \omega L$ (relevant om man använder $\tan^{-1}$ funktionen för att få ut fasvinkeln).

b) [1p] Bestäm effekten (tidsgenomsnitt, d.v.s. 'aktiveffekt' i växelströmsterminologi) som levereras till motståndet R.

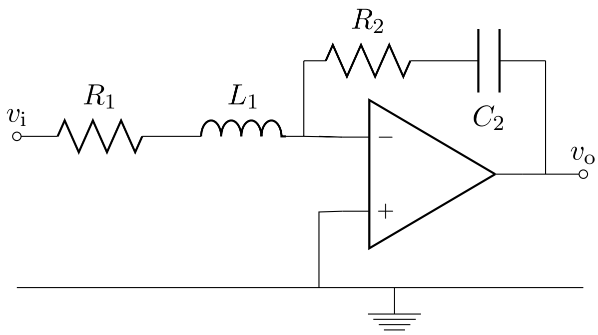


5) [5p]

a) [3p] Bestäm nätverksfunktionen $v_{\mathrm{o}}(\omega)/v_{\mathrm{i}}(\omega)$ av kretsen. Visa att den kan skrivas i formen

$$H(\omega) = \frac{v_{\mathrm{o}}(\omega)}{v_{\mathrm{i}}(\omega)} = \frac{-(1 + \mathrm{j}\omega/\omega_1)}{(\mathrm{j}\omega/\omega_2)\,(1 + \mathrm{j}\omega/\omega_3)}.$$

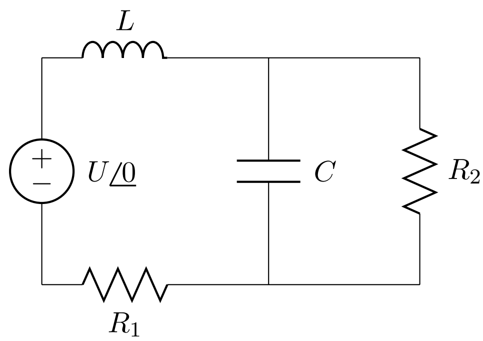
b) [2p] Skissa ett Bode amplituddiagram av funktionen $H(\omega)$. Anta att $\omega_1 \ll \omega_2 \ll \omega_3$. Markera viktiga punkter och lutningar.



6) [5p] Källan ger en växelspänning med effektivvärdet U och vinkelfrekvens ω. Bara U, ω, L och R_1 är kända storheter.

a) [4p] Välj R_2 och C för att maximera effekten som levereras till motståndet R_2.

b) [1p] Vad är den maximala effekten till R_2? Uttryck den som en funktion av de kända storheterna.



Slut. Men slösa inte eventuell återstående tid: *kolla och dubbelkolla svaren!*

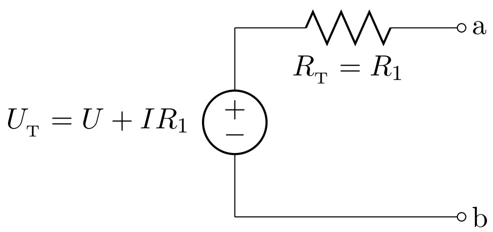
Solutions (EI1102 / EI1100, 2015-03-17)

Q1

a) With terminals a-b in the open-circuit condition, all the current I must pass through R_1. The open-circuit voltage $u_{\text{ab,oc}}$ is therefore $U + IR_1$.

The Thevenin resistance at a-b can be found most easily by setting the sources to zero and finding the resistance between terminals 'a' and 'b': this method is possible as there are no dependent sources in the circuit. In that case, the current-source branch is open-circuit (zeroed current source), and the voltage-source short-circuits R_3. The result is simply R_1.

It is important to show how the terminals a-b are related to the poles of the voltage source in the Thevenin equivalent. This is conveniently done by a diagram:



The position of the voltage-source and resistor in the equivalent are not important, but they must be in series and the voltage source + and − terminals must connect to the right choice of the output terminals 'a' and 'b' so that the equivalent has the correct sign of its voltage.

b) The maximum power output from the two-terminal circuit — or its equivalent — is obtained when the current and voltage are half their short-circuit and open-circuit values, respectively. (For a resistive load, this is achieved when $R_{\text{load}} = R_{\mathrm{T}}$.) Hence, the maximum power is

$$P = \frac{u_{\text{oc}}}{2} \cdot \frac{i_{\text{sc}}}{2} = \frac{U_{\mathrm{T}}}{2} \cdot \frac{U_{\mathrm{T}}}{2R_{\mathrm{T}}} = \frac{U + IR_1}{2} \cdot \frac{U + IR_1}{2R_1} = \frac{U_{\mathrm{T}}^2}{4R_{\mathrm{T}}} = \frac{(U + IR_1)^2}{4R_1}.$$

The final answer for maximum power should be expressed as $\frac{(U+IR_1)^2}{4R_1}$, as U_{T} and R_{T} are not known quantities.

Note that R_2 and R_3 were irrelevant to the answers. This could directly have been seen (and thus the circuit simplified at the start), as R_2 is directly in series with a current source, and R_3 is directly in parallel with a voltage source. As long as we only consider things outside the region of those sources and resistors, the resistors will not make any difference: they do however affect how much power the directly-coupled sources are supplying.

Q2

a) Power into R_2 is $I_2^2 R_2$ (it's series with the current source).

b) Power into R_3 is $\dfrac{(U_1 + U_2)^2}{R_3}$ (its voltage is directly set by the pair of voltage sources).

c) Power into R_1 is $(I_1 + I_2)^2 \, R_1$ (KCL at the bottom node shows that the current in R_1 is $I_1 + I_2$).

d) The source's voltage is already defined. Its current needs to be found in order to find the power. The current in source U_1, if we define it coming *out* of the + pole ('active convention') is fairly easy to find, since it is the sum of currents in R_3 and R_4, which are determined by just U_1 and U_2. Hence this current is $\frac{(U_1+U_2)(R_3+R_4)}{R_3R_4}$. The power coming out from the source into the circuit is the product of this current and the source voltage, $U_1 \frac{(U_1+U_2)(R_3+R_4)}{R_3R_4}$.

e) The source's current is already defined. So its voltage needs to be found in order to find the power. Let's define the voltage on source I_2 so that the reference + side is where the defined current comes out ('active convention' again). This requires some effort. KVL is the basic tool: but the smallest loop includes I_1 which also has an unknown voltage. So try the next loop, which goes I_2, R_2, R_1, U_2. Using KVL and inserting known values of voltages, we find the voltage across source I_2 must be $I_2R_2 + (I_1 + I_2)\,R_1 + U_2$. Then the power out from source I_2 into the circuit is $I_2^2\,(R_1 + R_2) + I_2I_1R_1 + I_2U_2$.

Q3

There is just one change in this circuit over all time: the switch closes at $t = 0$. We need to find the current in the inductor for all time after this change. This current is the *continuous variable* of the inductor, which means that immediately after the circuit changes, this current must still be the same as just before the change: $i(0^+) = i(0^-)$. (This does *not* have to be true for the *voltage* on the inductor, or for other quantities in the circuit in general!)

Before the switch closes, equilibrium can be assumed, because the circuit has been standing "a long time" (i.e. since $t = -\infty$). It is prudent to check also that the circuit has no special cases that could cause there not to be a constant-equilibrium solution: for example, a voltage source directly across an inductor, or a dependent source that results in a negative resistance seen by a reactive component. However, we take care to avoid such cases in exam questions!

In this equilibrium the inductor can be treated as a short-circuit: it has a steady current, so its voltage is zero, as $u(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t}$ for an inductor. Taking KVL around the loop of U, R_1, L and R_2, with zero voltage on L, we find $i(0^-) = \frac{U}{R_1+R_2}$.

When the switch closes, the branch of L and R_1 is connected to a short-circuit. All the other components thus become irrelevant to what happens around L. KVL around this loop gives that

$$L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + i(t)R_1 = 0.$$

The final value, $i(\infty)$ is zero: this can be seen by noticing that current in R_1 loses energy, but there is no source in this branch; or consider that the inductor L is connected to a Thevenin source with zero voltage and resistance R_1.

The solution of $i(t)$ for $t > 0$ is then $i(t) = \frac{U}{R_1+R_2}\mathrm{e}^{-tR_1/L}$. This could be found by solving the differential equation and using the initial condition of $i(0^+)$. Alternatively, it could be found by the 'quick method' using the initial value $i(0^+)$, the final value 0, and the time-constant ($\tau = L/R_1$).

Q4

a) Let's translate this time-domain current $I(t) = \hat{I}\sin(\omega t)$ to a phasor $I(\omega) = \hat{I}\underline{/0}$. (This implies we're using peak magnitude and sine-reference.)

By current division,

$$i_x(\omega) = I(\omega)\,\frac{\frac{1}{\mathrm{j}\omega C}}{R + \mathrm{j}\left(\omega L - \frac{1}{\omega C}\right)}.$$

In order to get back to a time-function, the magnitude and phase of this expression must be found. We could keep it in the above form, and find the magnitude and phase of the top and bottom parts, then combine these (polar-form calculation). Or we could try manipulating the expressions so that the only complex part is at the top or the bottom. The latter is probably the neatest choice here,

$$i_x(\omega) = \frac{\hat{I}\underline{/0}}{(1-\omega^2CL) + \mathrm{j}\omega CR},$$

from which the polar form is

$$i_x(\omega) = \frac{\hat{I}}{\sqrt{(1-\omega^2 CL)^2 + (\omega CR)^2}} \angle\!-\tan^{-1}\tfrac{\omega CR}{1-\omega^2 CL}.$$

Notice that the $\tan^{-1}$ expression depends on the condition that the question specified: $\frac{1}{\omega C} > \omega L$. This ensures that the $1-\omega^2 CL$ term is positive. If this term were instead negative, then the denominator in the inverse tangent expression would be negative, meaning that the resulting angle should be shifted by π to give the correct polar form.

Remembering that we used peak values and a sine reference when converting from time-functions to phasors, we convert back to time in the same way:

$$i_x(t) = \frac{\hat{I}}{\sqrt{(1-\omega^2 CL)^2 + (\omega CR)^2}} \sin\left(\omega t - \tan^{-1}\frac{\omega CR}{1-\omega^2 CL}\right).$$

b) The familiar $|i|^2 R$ formula for power into a resistor can be used. We must remember that we chose the phasor to be a peak value, so the power calculation must include the factor of $\frac{1}{2}$.

$$P_{\mathrm{R}} = \frac{1}{2} \cdot \frac{\hat{I}^2}{(1-\omega^2 CL)^2 + (\omega CR)^2} \cdot R$$

Q5

a) This is an inverting amplifier. If we combine R_1 with L_1 as impedance Z_1, and R_2 with C_2 as impedance Z_2, then the familiar result is that $\frac{v_{\mathrm{o}}(\omega)}{v_{\mathrm{i}}(\omega)} = \frac{-Z_2}{Z_1}$. Writing the full expressions for the impedances, in terms of the known quantities,

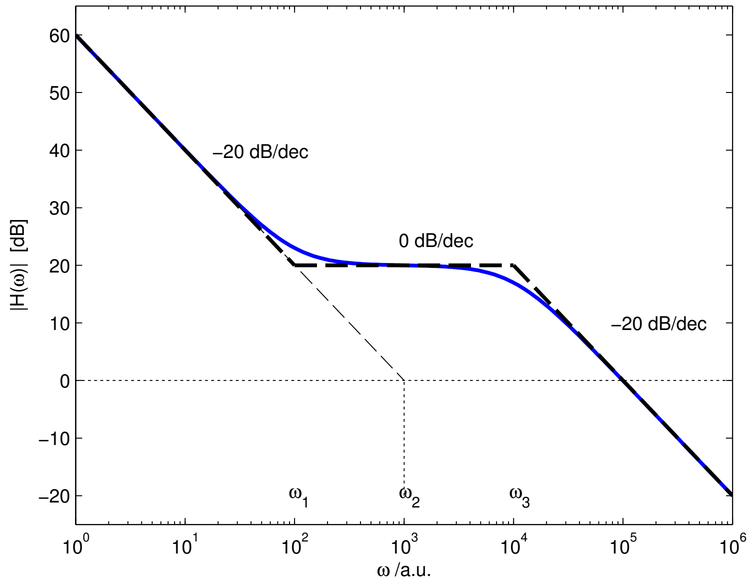
$$H(\omega) = \frac{v_{\mathrm{o}}(\omega)}{v_{\mathrm{i}}(\omega)} = -\frac{R_2 + \frac{1}{\mathrm{j}\omega C_2}}{R_1 + \mathrm{j}\omega L_1} = \frac{-\left(1 + \mathrm{j}\omega C_2 R_2\right)}{\mathrm{j}\omega C_2 R_1 \left(1 + \mathrm{j}\omega L_1/R_1\right)}.$$

By defining $\omega_1 = \frac{1}{C_2 R_2}$, $\omega_2 = \frac{1}{C_2 R_1}$ and $\omega_3 = \frac{1}{L_1/R_1}$, this has the requested form.

You might prefer to remove the minus-sign and move the j from the bottom to the top of this expression for $H(\omega)$. That helps to show the phase, but it no longer has the canonical first-order network-function 'building blocks' of $\mathrm{j}\omega/\omega_x$ etc.

b) The following plot indicates the expected shape.

The necessary parts (for getting full points) are: the straight lines that are shown here as thick dashed lines; the markings of where the gradient is $-20\,\mathrm{dB/decade}$; the frequencies ω_1 and ω_3 at the points where the gradient changes; and the flat part should be *above* $0\,\mathrm{dB}$, because the initial slope would have passed through $0\,\mathrm{dB}$ at $\omega = \omega_2$ (which is $> \omega_1$). Other numeric details shown here are just one possible example within the given criteria of $\omega_1 \ll \omega_2 \ll \omega_3$; these numbers are not required in the exam solution.



In this example the curved blue line is the actual function, without the asymptotic approximation of the Bode plot. The term $\frac{1}{\mathrm{j}\omega/\omega_2}$ is at 0 dB when $\omega = \omega_2$: the relation of ω_1 to ω_2 is what determines the level of the flat (0 dB/dec) part of the plot, because it is when $\omega > \omega_1$ that the $(1 + \mathrm{j}\omega/\omega_1)$ starts to cancel the downward slope of the $\frac{1}{\mathrm{j}\omega/\omega_2}$ term.

Q6

a) This is a quite typical ac maximum-power problem, with a Thevenin source and a parallel load impedance. For maximum power transfer, we need to make the load impedance equal the complex conjugate of the source impedance.

It would have been very easy if the question had a Thevenin source with a load made of series components, or a Norton source with a load made of parallel components. Here, a little more work is needed. It's definitely best to start by equating *admittances*: that's because the two unknowns (R_2 and C) are in parallel, so they are the real and imaginary parts of the admittance. Then it's easy to equate each one separately with the real or imaginary part of the total impedance.

$$Y_{\mathrm{src}}^* = \left(\frac{1}{R_1 + \mathrm{j}\omega L}\right)^* = \frac{R_1 + \mathrm{j}\omega L}{R_1^2 + \omega^2 L^2} = Y_{\mathrm{load}} = \frac{1}{R_2} + \mathrm{j}\omega C.$$

The real and imaginary parts can then be equated directly to find R_2 and C,

$$R_2 = \frac{R_1^2 + \omega^2 L^2}{R_1}, \quad C = \frac{L}{R_1^2 + \omega^2 L^2}.$$

b) The quickest way to find the [active-]power delivered to the load in the maximum-power condition, in this question, is to find the maximum power possible from the source. (This choice is convenient in our case, because the source components have the known values. We could instead calculate using the load and source values, but then there would be further arithmetic needed to get the solution in terms of just the known quantities of U, R_1 and L.)

Considering an impedance load of $Z_{\mathrm{load}} = Z_{\mathrm{src}}^*$, we think of the loop of the voltage-source U, then $Z_{\mathrm{src}} = R_1 + \mathrm{j}\omega L$, then load $Z_{\mathrm{load}} = R_1 - \mathrm{j}\omega L$. The total impedance in this loop is just $2R_1$ (which is the same as the dc maximum power situation if the reactive components are ignored).

The power to the load is then $P_{\max} = \frac{U^2}{4R_1}$.
(The 4 is because the current is ${}^{U}\!/_{2R_1}$, and the power to the load is $|i|^2 R_1$; there are a few other ways to

reason to the same conclusion. If the phasor $U\underline{/0}$ had been a peak value instead of rms, then a further $1/2$ would have been needed.)

A longer way to get the solution would be to insert the calculated values for R_2 and C (from part 'a'), then to do some simplification. This simplification would really just be undoing the work in part 'a', to come back to the load and source impedances being complex conjugates. We could find the voltage across R_2 by voltage division, as

$$u_{\text{load}} = U \frac{\frac{1}{\frac{1}{R_2}+\mathrm{j}\omega C}}{\frac{1}{\frac{1}{R_2}+\mathrm{j}\omega C} + R_1 + \mathrm{j}\omega L},$$

then express the power in this resistor as u_{load}^2/R_2, and substitute the values found for R_2 and C in part 'a'. After simplification, this comes out to the above result of $\frac{U^2}{4R_1}$. But it's preferable to make a wiser choice from the start, thereby avoiding the extra work.