KTH EI1102 / EI1100 Elkretsanalys Omtenta 2015-06-11 kl 14–19

Tentan har 6 tal i 2 delar: tre tal i del A (15p), tre i del B (15p).

Hjälpmedel: Ett A4-ark (båda sidor) med studentens egna anteckningar på valfritt sätt: handskrivet eller datorutskrift; text eller diagram; stor eller liten textstorlek, . . .).

Om inte annan information anges i ett tal, ska: komponenter antas vara idéala; angivna värden av komponenter (t.ex. R för ett motstånd, U för en spänningskälla, k för en beroende källa) antas vara *kända* storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller spänningskälla) antas vara *okända* storheter. Lösningar ska uttryckas i kända storheter, och förenklas. Var tydlig med diagram och definitioner av variabler.

Tips: Dela tiden mellan talen. Senare deltal brukar vara svårare att tjäna poäng på: fastna inte på dessa. Det hjälper, ofta, att rita om ett diagram för olika tillstånd eller med ersättningar eller borttagning av delar som inte är relevanta för det sökta värdet. Då blir kretsen ofta mycket lättare att tänka på och lösa. Kontrollera svarens rimlighet genom t.ex. dimensionskoll eller alternativ lösningsmetod.

Godkänd tenta kräver minst 25% i del A, 25% i del B, och 50% i genomsnitt (båda delar). Betyget räknas från summan över båda delar, med gränser (%) av 50 (E), 60 (D), 70 (C), 80 (B), 90 (A).

Examinator: Nathaniel Taylor

Del A. Likström och Transient

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1) [5p]
Bestäm de följande:

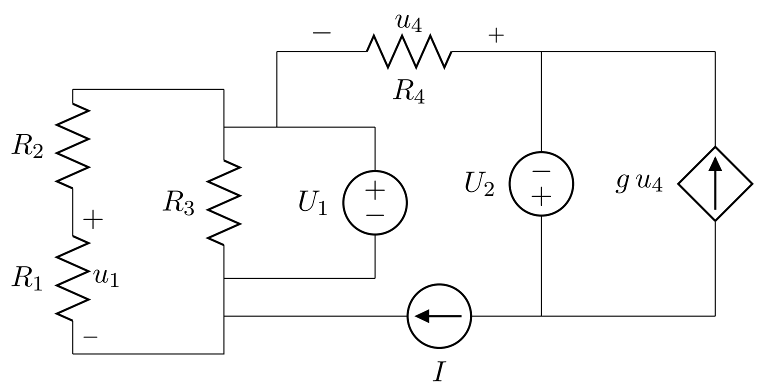
a) [1p] Effekten levererat till R_3.

b) [1p] Spänningen u_1 över R_1.

c) [1p] Effekten levererat till R_4.

d) [1p] Spänningen u_4 över R_4.

e) [1p] Effekten levererat från källan U_2.



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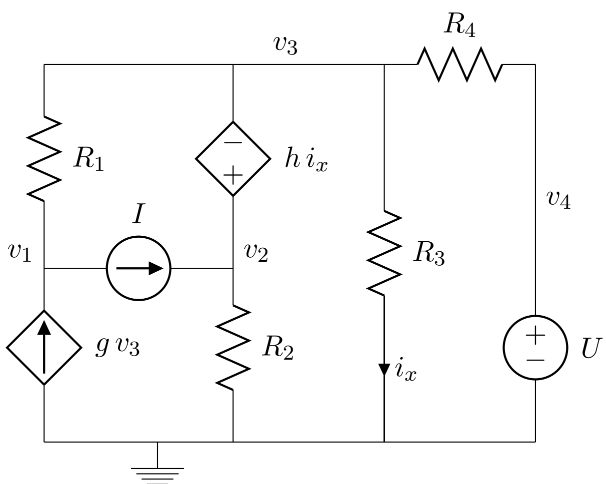
2) [5p]

Använd nodanalys för att skriva ekvationer som skulle kunna lösas för att få ut de markerade nodpotentialerna v_1, v_2, v_3, v_4.

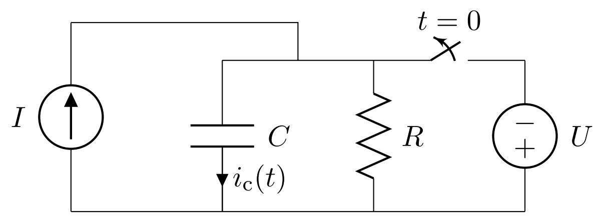
Du behöver bara visa att du kan översätta från kretsen till ekvationerna: du *måste inte* lösa eller förenkla ekvationerna.

Som vanligt är det komponentvärdena R_1, I, g, h o.s.v. som är kända, medan de markerade storheterna v_1, i_x, o.s.v. är okända.



3) [5p]

Bestäm $i_c(t)$, för $t > 0$.

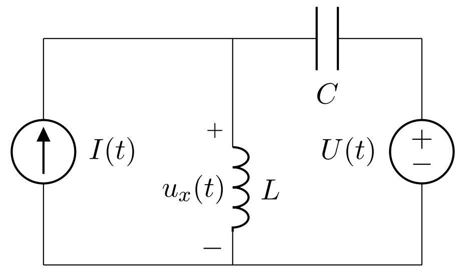


Del B. Växelström

4) [5p]

Källornas värden är: $U(t) = \hat{U}\sin(\omega t)$, $\quad I(t) = \hat{I}\cos(\omega t)$.

Bestäm $u_x(t)$.



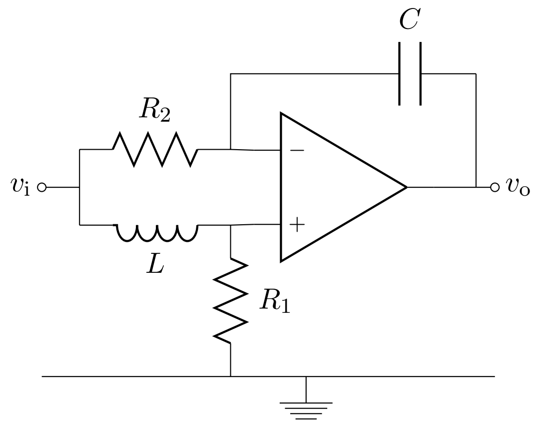
5) [5p]

a) [2p] Bestäm nätverksfunktionen $H(\omega) = \frac{v_o(\omega)}{v_i(\omega)}$ av kretsen.

b) [1p] Visa att $H(\omega)$ i deltal 'a' kan skrivas som

$$H(\omega) = \frac{k}{1 + j\omega/\omega_x},$$

där k och ω_x är positiva reella tal (vid antagandet att komponentvärdena också är det).

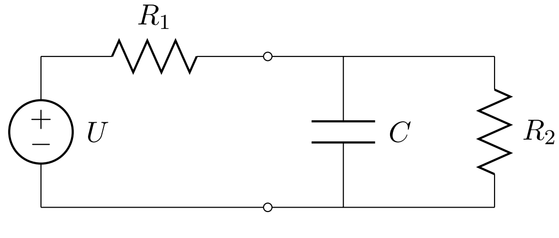


c) [2p] Skissa ett Bode amplituddiagram av funktionen $H(\omega)$ från deltal 'b'. Markera viktiga punkter och lutningar.

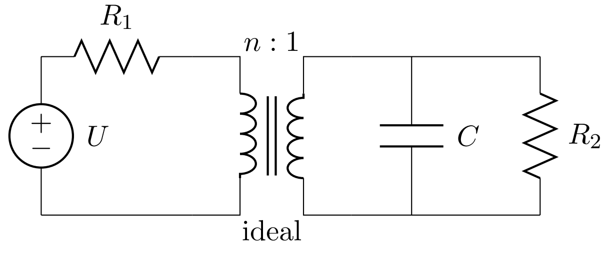
6) [5p]

Källan ger en växelspänning med vinkelfrekvens ω, beskriven av fasvektorn U.

a) [3p] Betrakta den övre kretsen. Komponentvärdena R_2 och C är okända; de andra värdena är kända storheter. Välj R_2 och C för att maximera effekten som levereras till motståndet R_2.



b) [2p] Betrakta nu den nedre kretsen, där en ideal transformator med relativ antal varv $n : 1$ används i stället för direktkoppling. Välj R_2 och C med samma villkoren som i deltal 'a', men nu med transformatorn. Kvoten n är känd.



Solutions (EI1102/EI1100, VT15, 2015-06-11)

Q1

a) Power into R_3 is U_1^2/R_3, as the resistor is parallel with the voltage source U_1, which fixes its voltage and makes all the other components irrelevant to the solution.

b) Voltage u_1 is $U_1 \dfrac{R_1}{R_1 + R_2}$, as the series pair R_1 and R_2 is in parallel with the voltage source U_1 which fixes their voltage and makes all other components than these three irrelevant to u_1.

c) Power into R_4 is $I^2 R_4$. This resistor is in series with the independent current source I! It may not be obvious immediately, but it can be seen by considering the whole region of $\{ R_1, R_2, R_3, U_1 \}$: KCL says that the current in from I has to pass out into R_4. Alternatively, consider KCL on the right-hand side of the circuit, for voltage source U_2 and the dependent current source $g\,u_4$.

d) By seeing that the current from right to left in R_4 must be $-I$, we get $u_4 = -IR_4$.

e) The current through the voltage source U_2 needs to be found, in order to find what power this source delivers. Let's define the source's current as i_2, out from the '+' terminal. Then by KCL at the node below U_2, $i_2 = I + gu_4$. Substituting the u_4 from question 'e' (above), the KCL expression becomes $i_2 = I\,(1 - gR_4)$. The power out from source U_2 is therefore $U_2 I\,(1 - gR_4)$.

Q2

We'll show the "extended nodal analysis" (which I've sometimes called the simple method, because it has few rules although it generates longer equation systems).

First, KCL at every node except ground. We'll define the currents in the voltage sources as i_α (in U) and i_β (in the dependent voltage source), both being into the '+' terminal.

$$\begin{aligned}
\mathrm{KCL}(1): \quad 0 &= \frac{v_1 - v_3}{R_1} + I - g\,v_3 && (1)\\
\mathrm{KCL}(2): \quad 0 &= \frac{v_2}{R_2} - I + i_\beta && (2)\\
\mathrm{KCL}(3): \quad 0 &= \frac{v_3 - v_1}{R_1} - i_\beta + \frac{v_3}{R_3} + \frac{v_3 - v_4}{R_4} && (3)\\
\mathrm{KCL}(4): \quad 0 &= \frac{v_4 - v_3}{R_4} + i_\alpha. && (4)
\end{aligned}$$

Then, include the relations of node potentials that the voltage sources determine,

$$\begin{aligned}
v_4 &= U && (5)\\
v_2 - v_3 &= h\,i_x. && (6)
\end{aligned}$$

Finally, define controlling variables of dependent sources in terms of already defined quantities. In this case, one source had a node potential as its controlling variable, so it needs no equation (it would be just '$v_3 = v_3$').

$$i_x = \frac{v_3}{R_3} \qquad (7)$$

We will leave other methods, such as the supernode approach, as 'an exercise for the reader'.

Q3

After the switch opens, the current I of the current source must split between the capacitor and resistor. Let's define the voltage across the capacitor (with the positive reference upwards) as u_c.

As the resistor and capacitor are in parallel, the current in the resistor is known as $\frac{u_\mathrm{c}(t)}{R}$.
By KCL, we see that

$$i_\mathrm{c}(t) = I - \frac{u_\mathrm{c}(t)}{R}.$$

The constitutive equation for a capacitor is $i_\mathrm{c}(t) = C\frac{\mathrm{d}u_\mathrm{c}(t)}{\mathrm{d}t}$, which allows a differential equation in u_c to be written as

$$C\frac{\mathrm{d}u_\mathrm{c}(t)}{\mathrm{d}t} = I - \frac{u_\mathrm{c}(t)}{R}.$$

It would also have been possible to write a differential equation in another variable such as i_c or i (which is the variable we're ultimately solving for); we have chosen to use the continuous variable, since this makes it easy to handle the initial condition.

The initial condition for u_c has to be found from the equilibrium before the switch opened. With the switch closed ($t < 0$) the voltage source was connected in parallel with the resistor and capacitor. The capacitor voltage must therefore have been $u_\mathrm{c}(0^-) = -U$, by KVL. By continuity, $u_\mathrm{c}(0^+) = -U$.

The above differential equation and initial condition can be solved for $u_\mathrm{c}(t)$ during $t > 0$. The alternative method is to use the initial value ($u_\mathrm{c}(0^+) = -U$), the final value ($u_\mathrm{c}(\infty) = IR$), and the time-constant given by the capacitor and the equivalent source that it sees ($\tau = RC$ here) to write the solution directly. Either way,

$$u_\mathrm{c}(t) = IR - (U + IR)\,\mathrm{e}^{-\frac{t}{CR}}.$$

We were actually looking for $i_\mathrm{c}(t)$, the current through the capacitor. This has already been seen to be $i_\mathrm{c}(t) = C\frac{\mathrm{d}u_\mathrm{c}(t)}{\mathrm{d}t}$, so we can write

$$i_\mathrm{c}(t) \;=\; C\frac{\mathrm{d}}{\mathrm{d}t}\left(IR - (U + IR)\,\mathrm{e}^{-\frac{t}{CR}}\right) \;=\; \left(\frac{U}{R} + I\right)\,\mathrm{e}^{-\frac{t}{CR}}.$$

Q4

Let's take a sine reference, so that $U(t) = \hat{U}\sin(\omega t)$ becomes $U(\omega) = \hat{U}\underline{/0}$.
Then $I(t) = \hat{I}\cos(\omega t)$ becomes $I(\omega) = \hat{I}\underline{/\pi/2} = \mathrm{j}\hat{I}$.

The phasor equation for KCL in the node above L, omitting for neatness the '(ω)' after the phasors, is

$$-I + \frac{u_x}{\mathrm{j}\omega L} + \mathrm{j}\omega C\,(u_x - U) = 0,$$

from which the sought voltage is

$$u_x(\omega) = \frac{I + \mathrm{j}\omega CU}{\mathrm{j}\left(\omega C - \frac{1}{\omega L}\right)} = \frac{\hat{I} + \omega C\hat{U}}{\omega C - \frac{1}{\omega L}}.$$

This expression has no imaginary parts. Its phase-angle is either 0 or π, depending on the sign of the denominator. (If a different reference, such as $\cos(\omega t)$ or $\sin(\omega t + \pi/2)$ had been used, then the phasor $u_x(\omega)$ would not have zero angle, but it would still have the special feature of being a particular angle or that angle plus π.)

One way of writing the time-function is

$$u_x(t) = \frac{\hat{I} + \omega C\hat{U}}{\omega C - \frac{1}{\omega L}}\,\sin(\omega t)\,.$$

This looks rather different from what we often see when solving ac circuits and writing their time functions. Normally we have phase-angles that can vary over a range; then we have written time-functions in the form $f(t) = A\sin(\omega t + \phi)$ (or with cos instead), where A is positive real. Some square-roots are usually needed for finding the magnitude A, and perhaps an inverse tangent for finding ϕ. In the case above, we have instead allowed the term before the sin function to be positive or negative, to make the phase-angle change by π when the difference between ωC and $1/\omega L$ changes sign. The phasor was already purely real, so the square-roots and inverse tangents were not needed. This choice seemed the simplest way to get the right behaviour. Note that when $\omega C = 1/\omega L$ there is a resonance at which u_x has unbounded magnitude.

Q5

a) The potentials of the two opamp inputs are expected to be the same, due to the negative feedback. Let this potential be v. Then, from KCL separately at the two inputs, we get two equations,

$$\frac{v - v_{\rm i}}{{\rm j}\omega L} + \frac{v}{R_1} = 0, \qquad \text{and} \qquad \frac{v - v_{\rm i}}{R_2} + (v - v_{\rm o})\,{\rm j}\omega C = 0.$$

Eliminating v and rearranging,

$$\frac{v_{\rm o}}{v_{\rm i}} = H(\omega) = \frac{(1 + {\rm j}\omega C R_2) - (1 + {\rm j}\omega L/R_1)}{{\rm j}\omega C R_2\,(1 + {\rm j}\omega L/R_1)}.$$

This can be further simplified as shown in part 'b)'.

b) Simplification of the result from part 'a)' gives,

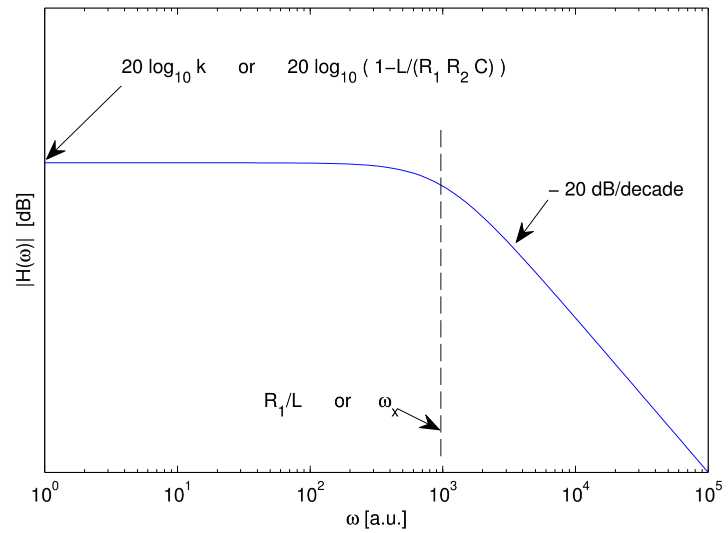
$$H(\omega) = \frac{1 - \frac{L}{CR_1R_2}}{1 + {\rm j}\omega L/R_1},$$

which can be written in the requested form of $\frac{k}{1+{\rm j}\omega/\omega_x}$ by setting

$$k = 1 - \frac{L}{CR_1R_2} \qquad \text{and} \qquad \omega_x = \frac{R_1}{L}.$$

This is a quite surprisingly simple function: swapping of the positions of the same components in this circuit can result in more complicated Bode plots.

c)



This is an unusually easy Bode amplitude plot. It is just the curve of a single pole, shifted vertically by k (in dB). The values k and ω_x could be used, or their equivalent in terms of circuit quantities such as R_1/L. In case you wonder: 'a.u.' means 'arbitrary units'. Normally the hand-written Bode plot would use straight lines instead of showing the curve around ω_x.

Q6

a) As C and R_2 are able to be chosen, while other components are fixed, it is convenient to see this maximum power question as having a source of U and R_1, a load of C and R_2.

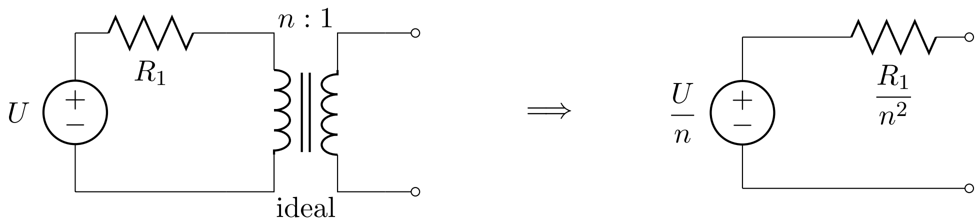
Maximum power to the load is attained when the load impedance is the complex conjugate of the source impedance. In this case, the source impedance is pure resistance: we need $R_2 = R_1$ and $C = 0$.

This can be shown more formally by writing

$$Z_{\text{load}} = \frac{\frac{R}{\mathrm{j}\omega C}}{R + \frac{1}{\mathrm{j}\omega C}} = Z^*_{\text{source}} = R_1^* = R_1.$$

b) The only difference now is the $n : 1$ transformer between source and load. Probably the simplest way to start is to replace the source and transformer by an equivalent source that represents what the load 'sees' at the terminals of the right-hand side of the transformer.

If we disconnect the load from the transformer, the open-circuit voltage of the transformer's right-hand coil is U/n. The resistance R_1 is equivalent to a resistance R_1/n^2 on the other side of the transformer: this is a commonly used substitution, and can be derived by considering the ratios of primary to secondary voltage ($n : 1$) and current ($1 : n$). Thus, the following substitution can be made:



By the same reasoning as in part 'a)', we need $R_2 = R_1/n^2$ and $C = 0$.
As usual with maximum power calculations, the source voltage was not relevant to what load impedance should be chosen, although it would affect how large the maximum power is.