KTH EI1102 / EI1100 Elkretsanalys Omtenta   2016-10-28 kl 08–13

Hjälpmedel: Ett A4-ark (båda sidor) med studentens egna anteckningar på valfritt sätt: handskrivet eller datorutskrift; text eller diagram; stor eller liten textstorlek, ... Det behöver *inte* lämnas in.

Tentan har 6 tal i 2 delar: tre tal i del A (15p), tre i del B (15p).

Godkänd kräver minst 25% i del A, 25% i del B, och 50% i genomsnitt över A och B.
Betygsgränserna (%) är 50 (E), 60 (D), 70 (C), 80 (B), 90 (A).

Om inte annan information anges i ett tal, ska: komponenter antas vara idéala; angivna värden av komponenter (t.ex. R för ett motstånd, U för en spänningskälla, k för en beroende källa) antas vara **kända** storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller en spänningskälla) antas vara **okända** storheter.

Lösningar ska uttryckas i kända storheter, och förenklas. Var tydlig med diagram och definitioner.

Dela tiden mellan talen — senare deltal brukar vara svårare att tjäna poäng på . . . fastna inte!

Kontrollera svarens rimlighet genom t.ex. dimensionskoll eller alternativ lösningsmetod.

Lycka till!   Nathaniel Taylor (073 949 8572)

Del A. Likström och transient

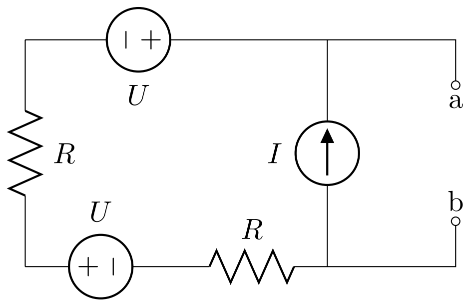
1) [5p]

a) [1p] Vilken spänning har pol 'a' relativ till pol 'b' här?

b) [1p] Vilken ström passar från pol 'a' till pol 'b' om dessa poler kortslutas?

c) [2p] Bestäm Nortonekvivalenten av kretsen, med avseende på polerna a-b. (Lösningarna ovan kan hjälpa.)

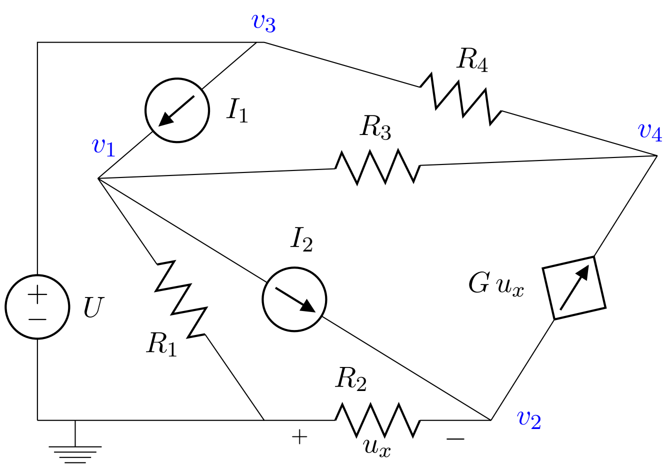
d) [1p] Bestäm den största effekten som kan fås ut från kretsen mellan polerna a-b.



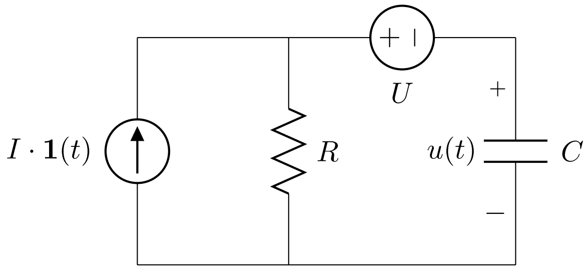
2) [5p]

Använd nodanalys för att skriva ekvationer som skulle kunna lösas för att få ut de markerade potentialerna v_1, v_2, v_3, v_4.

Du behöver bara visa att du kan översätta från kretsen till ekvationerna: du *måste inte* lösa eller förenkla ekvationerna.



3) [5p]

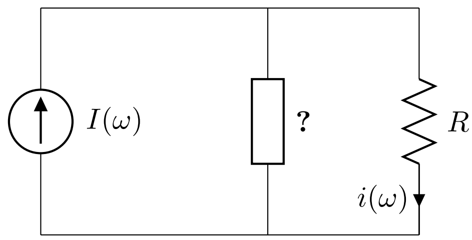


Bestäm $u(t)$ for $t > 0$.

Obs. $\mathbf{1}(\,)$ är enhetsstegfunktionen.

Del B. Växelström

4) [5p]



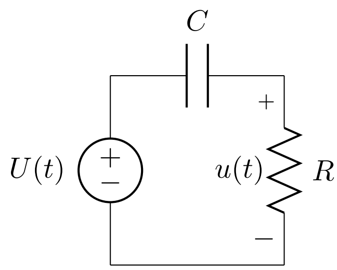
a) [3p] Skissa Bode amplituddiagram av nätverksfunktionen

$$H(\omega) = \frac{\mathrm{j}\omega/\omega_1}{1 + \mathrm{j}\omega/\omega_2}.$$

Antag att $\omega_1 = \omega_2$.
Använd dB-skalan, och markera viktiga punkter och lutningar.

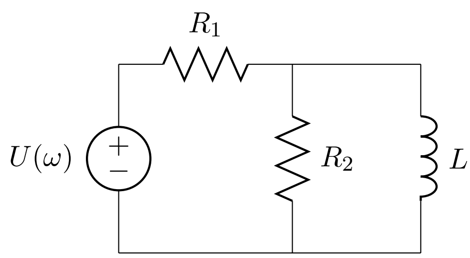
b) [2p] I kretsen ovan, är det en kondensator eller en spole som måste användas som komponenten markerad med **?** för att nätverksfunktionen $i(\omega)/I(\omega)$ ska ha samma formen som $H(\omega)$ från deltal 'a'? Visa tydlig motivering till svaret.

5) [5p]



En växelspänningskälla $U(t) = \hat{U} \sin\left(\omega t + \frac{\pi}{4}\right)$ matar kretsen.
Bestäm tidsfunktionen $u(t)$, genom växelströmsanalys ('jω'-metoden).

6) [5p]



Källan är en växelspänningskälla med vinkelfrekvens ω och amplitude $\hat{U}$ (toppvärde). Vilken aktiveffekt och skenbareffekt försörjs av källan?

Slut. *Men slösa inte eventuell återstående tid: kolla och dubbelkolla svaren.*

Solutions (EI1102 / EI1100 Omtenta HT16, 2016-10-28)

Q1.

a) With terminals a-b open-circuit, the current I passes through the resistors. KVL around the outer loop gives $u_{\text{ab}} = IR + U + IR + U$. Simplified, $u_{\text{ab}} = 2(U + IR)$.

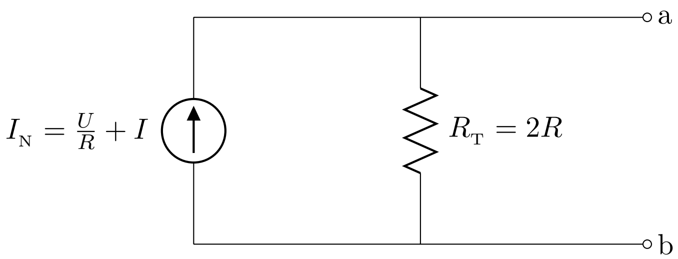
b) With a-b short-circuited, the current i_{ab} between the terminals is the sum of the short-circuit currents of the two branches, which is $i_{\text{ab}} = I + \frac{2U}{2R}$. The short-circuit current is therefore $i_{\text{ab}} = \frac{U}{R} + I$.

c) To determine the Norton equivalent, we need to know the Norton resistance and current.

The Norton current is the same as the short-circuit current, which was found in the previous part-question: $I_{\text{N}} = \frac{U}{R} + I$.

The Norton resistance is the ratio of open-circuit voltage to short-circuit current. For terminals a-b in the given circuit, those quantities have both been found in the previous part-questions.

$$R_{\text{N}} = \frac{u_{\text{ab}}}{i_{\text{ab}}} = \frac{2(U + IR)}{\frac{U}{R} + I} = \frac{2(U + IR)R}{U + IR} = 2R.$$



We could instead have found the Norton resistance for this circuit with no dependent sources, by simply "setting the independent sources to zero" and simplifying the resulting circuit of resistors.

c) The maximum possible power out from the circuit occurs when the terminal voltage is half of its open-circuit value. Equivalently, we can say it occurs when the current is half of its short-circuit value (this is true precisely when the voltage is half of its open-circuit value).

Hence this maximum power is

$$\frac{u_{\text{oc}}}{2} \cdot \frac{i_{\text{sc}}}{2} = \frac{I_{\text{N}}}{2} \cdot \frac{I_{\text{N}} R_{\text{T}}}{2} = \frac{I_{\text{N}}^2 R_{\text{T}}}{4}.$$

Putting in the values for our circuit,

$$\frac{I_{\text{N}}^2 R_{\text{T}}}{4} = \frac{\left(\frac{U}{R} + I\right)^2 2R}{4} = \frac{\left(\frac{U}{R} + I\right)^2 R}{2}.$$

Q2.

This solution uses extended nodal analysis ("the simple way").
Many variations of a valid solution are possible, ranging from four to many equations.

Let's define the unknown current in the voltage source as i_α into the + terminal.

Write KCL (let's take outgoing currents) at all nodes except ground:

$$\text{KCL}(1): \quad 0 = \frac{v_1}{R_1} + \frac{v_1 - v_4}{R_3} + I_2 - I_1 \tag{1}$$

$$\text{KCL}(2): \quad 0 = \frac{v_2}{R_2} + G\,u_x - I_2 \tag{2}$$

$$\text{KCL}(3): \quad 0 = i_\alpha + I_1 + \frac{v_3 - v_4}{R_4} \tag{3}$$

$$\text{KCL}(4): \quad 0 = \frac{v_4 - v_1}{R_3} + \frac{v_4 - v_3}{R_4} - G\,u_x \tag{4}$$

The voltage source gives a further equation relating a pair of node potentials,

$$v_3 \quad = \quad U. \tag{5}$$

The dependent source has a 'controlling variable' u_x, which is defined in the circuit diagram as a marked voltage across resistor R_2, with positive reference side on the earth node. We have to define this controlling variable as an equation,

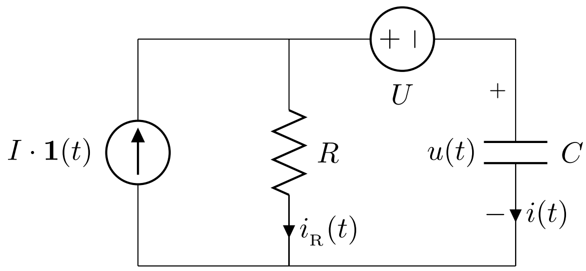
$$u_x \quad = \quad 0 - v_2. \tag{6}$$

Now there are 6 independent equations, in 6 unknown quantities.

THE ABOVE EQUATION-SYSTEM IS SUFFICIENT FOR A SOLUTION.

Q3.

Here is the circuit, with some further marked quantities. For $t \geq 0$ the current source has value I.



By KCL, the currents in the capacitor and resistor are related by

$$i(t) = I - i_{\mathrm{R}}(t).$$

The resistor's current can also be expressed in terms of KVL and Ohm's law, as

$$i_{\mathrm{R}}(t) = \frac{U + u(t)}{R}.$$

Putting these together,

$$i(t) = I - \frac{U}{R} - \frac{u(t)}{R}.$$

The relation of current and voltage in the capacitor is $i(t) = C\frac{\mathrm{d}u(t)}{\mathrm{d}t}$; note that these quantities have been defined according to the 'passive convention', so a negative sign is not needed. This relation can be inserted into the earlier equation to make a differential equation in the continuous variable $u(t)$ alone,

$$C\frac{\mathrm{d}u(t)}{\mathrm{d}t} = I - \frac{U}{R} - \frac{u(t)}{R},$$

which can be written in the form

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} + \frac{1}{RC}u(t) = \frac{I - U/R}{C},$$

having the general solution

$$u(t) = IR - U + k\,\mathrm{e}^{-t/RC}.$$

The value of k needs to be found from other knowledge: in this case the circuit gives enough information to find an initial condition. For $t < 0$ the current source has zero value, so it behaves as an open circuit. The equilibrium value of the resulting circuit is that $u(0^-) = -U$. By continuity (as the capacitor's voltage is its continuous variable, i.e. the energy-related one, that doesn't change instantaneously), this is also the voltage just after the current-source has a step change: $u(0^+) = u(0^-) = -U$. Inserting this, at the time $t = 0$ just after the step, noting that $\mathrm{e}^0 = 1$,

$$u(0^+) = -U = IR - U + k \qquad \Longrightarrow \qquad k = -IR$$

.

Now the specific solution can be written,

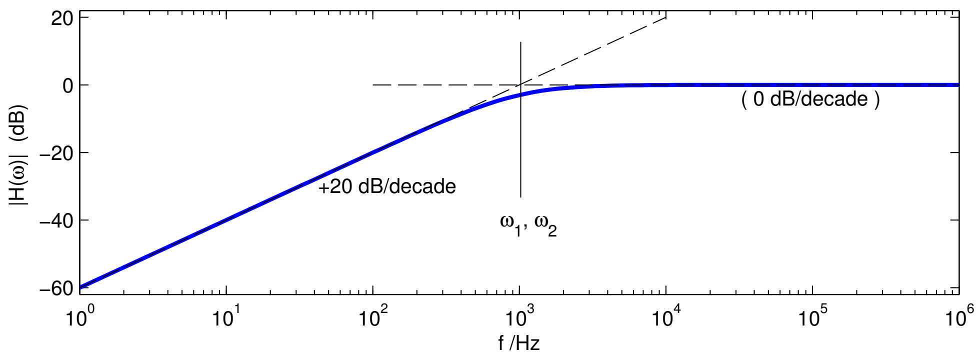
$$u(t) \;=\; IR - U - IR\,\mathrm{e}^{-t/RC} \;=\; -U + IR\left(1 - \mathrm{e}^{-t/RC}\right).$$

Checks:
Dimensionally this is ok, as RC is a time, and IR a voltage.
It gives a final value ($t \to \infty$) of $IR - U$, which makes sense given that all the current I passes down through R in equilibrium (capacitor is open-circuit).
It gives an initial value $-U$, as has already been determined.

Another method that could have been used would be to reduce the circuit other than the capacitor to a Thevenin (or Norton) equivalent, then solve the resulting circuit of a Thevenin voltage source and resistance when connected to a capacitor of known intitial conditions. This solution could be found by fitting an exponential function to a calculated initial value, final value and time-constant.

Q4.

a) This is a high-pass type of response.
A plot is shown below, for the case where $\omega_1 = \omega_2 = 2\pi \cdot 1\,\mathrm{kHz}$.



The main features to mark are the frequency ω_1 (and/or ω_2), the slopes (it's not essential to mark the zero slope), and the 0 dB level at $\omega \gg \omega_1$. As long as these details are marked, the numeric scales are not necessary: the frequency is arbitrarily chosen here. The classic asymptotic Bode plot shows straight lines, but it would be acceptable to show either the straight-line approximation or the actual, curved transition between the asymptotes.

b) If the component '**?**' is an inductor, the network function $i(\omega)/I(\omega)$ becomes as requested.
Let's be inventive by calling the inductor L: its impedance is then $\mathrm{j}\omega L$.
Current division gives

$$i(\omega) = I(\omega)\frac{\mathrm{j}\omega L}{R + \mathrm{j}\omega L},$$

in which we can divide the top and bottom both by R, and divide both sides by $I(\omega)$, to get the desired form.

$$\frac{i(\omega)}{I(\omega)} \;=\; \frac{\mathrm{j}\omega L/R}{R/R + \mathrm{j}\omega L/R} \;=\; \frac{\mathrm{j}\omega/({}^R\!/_L)}{1 + \mathrm{j}\omega/({}^R\!/_L)}.$$

By setting $\omega_1 = \omega_2 = R/L$ this agrees with the form in part 'a)'.

Q5.

We'll first convert the sinusoidal time-function of the source to a phasor. For convenience, let's choose a reference where a function $\sin(\omega t + {}^{\pi}/2)$ is represented by zero angle of the phasor, and the phasor's magnitude represents the peak value. By those choices, $U(\omega) = \hat{U}\angle 0$.

The capacitor's impedance is $Z_c = \frac{1}{j\omega C}$. (The resistor's is just R.)

By voltage division,

$$u(\omega) = \frac{R}{R+Z_c}U = \frac{R}{R+\frac{1}{j\omega C}}\hat{U} = \frac{j\omega CR}{1+j\omega CR}\hat{U}.$$

The final expression here would be good for plotting Bode diagrams. What we want, however, is an expression that easily can be converted into expressions for magnitude and angle of $u(\omega)$. For that purpose, the following is probably a little easier

$$u(\omega) = \frac{R}{R+Z_c}U = \frac{R}{R-j\frac{1}{\omega C}}\hat{U}.$$

The magnitude of this phasor $u(\omega)$ is

$$|u(\omega)| = \frac{\hat{U}R}{\sqrt{R^2+(\omega C)^{-2}}}$$

and the argument (phase), given that the top part of the above expression is purely real, is

$$\angle u(\omega) \;=\; \angle \frac{1}{R-j\frac{1}{\omega C}} \;=\; -\angle R-j\frac{1}{\omega C} \;=\; -\tan^{-1}\left(\frac{-1/(\omega C)}{R}\right) \;=\; \tan^{-1}\frac{1}{\omega CR}.$$

Now that we have expressed this phasor's magnitude and phase, we can write these as time-functions, using the same reference as before. Our reference was that a phasor of $1\angle 0$ would become $\sin(\omega t + {}^{\pi}/4)$. Using the actual values,

$$u(t) = \frac{\hat{U}R}{\sqrt{R^2+\left(\frac{1}{\omega C}\right)^2}}\ \sin\left(\omega t + {}^{\pi}/4 + \tan^{-1}\frac{1}{\omega CR}\right).$$

Q6.

If we define a current $i(\omega)$ out of the source's +-reference terminal (active convention), then the complex power supplied by the source is

$$S = \frac{1}{2}\cdot U(\omega)\cdot i(\omega)^*.$$

The factor 2 is needed if the phasors of source voltage $U(\omega)$ and current $i(\omega)$ are peak values rather than rms (effective) values.[1]

Active power is the real part, $P = \Re\{S\}$. Apparent power (skenbareffekt) is the magnitude $|S|$.

If we know the total impedance Z connected to the source, we can write the above as

$$S = \frac{1}{2}\cdot U(\omega)\cdot\left(\frac{U(\omega)}{Z}\right)^* = \frac{|U(\omega)|^2}{2Z^*} = \frac{\hat{U}^2}{2Z^*}.$$

The difficulty lies only in doing the calculation of impedance, and finding the real part and magnitude from the complex expressions.

[1] The question simply told us that the peak value of the voltage is $\hat{U}$, so we *could* choose to define $U(\omega)$ as a phasor based on the rms value, i.e. $\hat{U}/\sqrt{2}$. But we'll assume that we choose $U(\omega)$ to have a magnitude equal to the peak value of the sinusoidal time-waveform.

The impedance of the circuit driven by the source is

$$Z = R_1 + \frac{\mathrm{j}\omega L R_2}{R_2 + \mathrm{j}\omega L} = \frac{R_1(R_2 + \mathrm{j}\omega L) + \mathrm{j}\omega L R_2}{R_2 + \mathrm{j}\omega L} = \frac{R_1 R_2 + \mathrm{j}\omega L\,(R_1{+}R_2)}{R_2 + \mathrm{j}\omega L}.$$

From the above, we can write an expression for the complex power,

$$S \;=\; \frac{\hat{U}^2}{2Z^*} \;=\; \frac{\hat{U}^2}{2} \cdot \frac{R_2 - \mathrm{j}\omega L}{R_1 R_2 - \mathrm{j}\omega L\,(R_1{+}R_2)}.$$

Let's find the apparent power first, by finding the magnitude:

$$|S| \;=\; \frac{\hat{U}^2}{2} \cdot \frac{|R_2 - \mathrm{j}\omega L|}{|R_1 R_2 - \mathrm{j}\omega L\,(R_1{+}R_2)|} \;=\; \frac{\hat{U}^2}{2} \cdot \sqrt{\frac{R_2^2 + \omega^2 L^2}{R_1^2 R_2^2 + \omega^2 L^2\,(R_1{+}R_2)^2}}.$$

Now the active power, by finding the real part.
This takes a little more work to separate the real and imaginary parts.

$$\begin{aligned}
S &= P + \mathrm{j}Q \;\cdots \\
&= \frac{\hat{U}^2}{2} \cdot \frac{R_2 - \mathrm{j}\omega L}{R_1 R_2 - \mathrm{j}\omega L\,(R_1{+}R_2)} \\
&= \frac{\hat{U}^2}{2} \cdot \frac{(R_2 - \mathrm{j}\omega L)\,(R_1 R_2 + \mathrm{j}\omega L\,(R_1{+}R_2))}{R_1^2 R_2^2 + \omega^2 L^2\,(R_1{+}R_2)^2} \qquad \text{(multiply by complex conjugate of denominator)} \\
&= \frac{\hat{U}^2}{2} \cdot \frac{R_1 R_2^2 + \omega^2 L^2\,(R_1{+}R_2) + \mathrm{j}\omega L R_2^2}{R_1^2 R_2^2 + \omega^2 L^2\,(R_1{+}R_2)^2} \qquad \text{(expand numerator, re-group it)}
\end{aligned}$$

Now it's easy to take the real part alone:

$$P = \Re\{S\} = \frac{\hat{U}^2}{2} \cdot \frac{R_1 R_2^2 + \omega^2 L^2\,(R_1{+}R_2)}{R_1^2 R_2^2 + \omega^2 L^2\,(R_1{+}R_2)^2}.$$

The algebra was quite lengthy for such a simple-looking circuit.