KTH EI1120 Elkretsanalys (CENMI) TEN1 2020-06-05 kl. 08–13

Special Situation: This exam is taken remotely, under video supervision. Answers must be scanned and submitted before leaving the video meeting.

Permitted material: Beyond writing-equipment, up to three pieces of paper up to A4 size can be used, with free choice of content: handwritten, printed; small, large; text, diagram, image; one or both sides, etc. These papers do not need to be handed in with the exam.

Unless it is stated otherwise, the final answer to a question should be expressed in terms of the known quantities given in the question, and any clear simplifications should be done. Component values such as R for a resistor, U for an independent voltage source, or K for a dependent source, are assumed to be known quantities. Marked currents or voltages such as i_x are assumed to be definitions, not known quantities.

Clearly drawn and labelled diagrams are a good way to help yourself avoid mistakes, and to make clear to others what you are doing. By showing clearly your intermediate steps in a solution, you improve your chance of getting points even if the final result is wrong. You may write in Swedish or English; but we suggest that writing in either is seldom necessary if you make good use of diagrams and equations!

Determination of exam grade. Denote as A, B and C the available points from sections A, B and C of this exam: A=12, B=10, C=18. Denote as a, b and c the points actually obtained in the respective sections, and as a_k and b_k the points från KS1 and KS2, and as h the homework 'bonus'. The requirement for passing the exam (E or higher) is:

$$\frac{\max(a, a_\text{k})}{A} \geq 40\% \quad \& \quad \frac{\max(b, b_\text{k})}{B} \geq 40\% \quad \& \quad \frac{c}{C} \geq 40\% \quad \& \quad \frac{\max(a, a_\text{k}) + \max(b, b_\text{k}) + c + h}{A + B + C} \geq 50\%$$

The grade is then determined by the total including bonus, i.e. the last of the terms above: boundaries (%) are 50 (E), 60 (D), 70 (C), 80 (B), 90 (A). If the exam misses a pass by a small margin on just one criterion, a grade of Fx may be registered, with the possibility of completing to E by an extra task arranged later.

Special for the VT20 round: The optional project-task with up to 6 points substitutes for Question 9 in this exam if that gives an advantage. In selecting whether to use points from the exam or part-exams ('KS'), the selection will be done per question not just per section.

Nathaniel Taylor (08 790 6222)

Section A. Direct Current

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1) [4p]

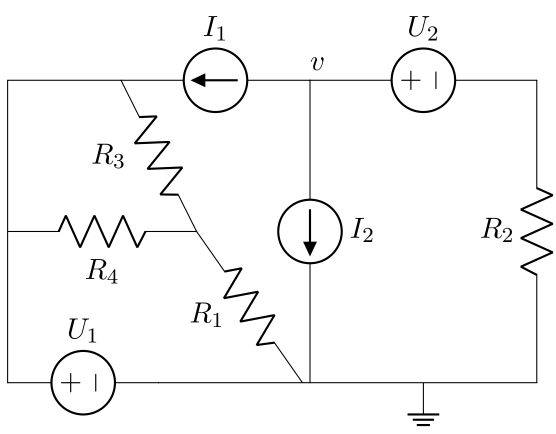
Determine:

a) [1p] the power from source U_2

b) [1p] the marked potential v

c) [1p] the power into resistor R_3

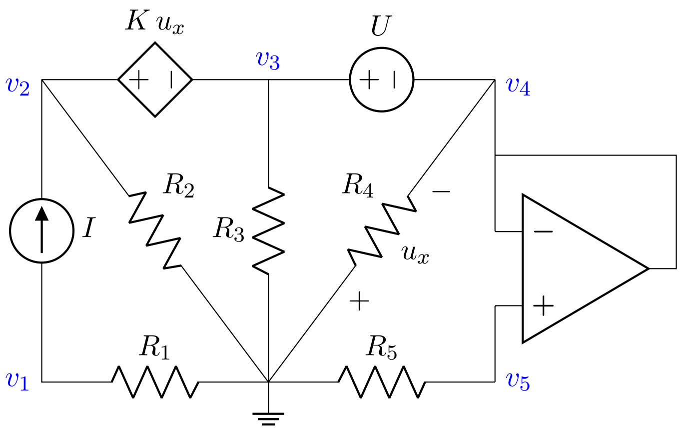
d) [1p] the power from source I_1



2) [4p]

Write equations that could be solved without further information to find the potentials v_1, v_2, v_3, v_4 and v_5 in this circuit in terms of the component values.

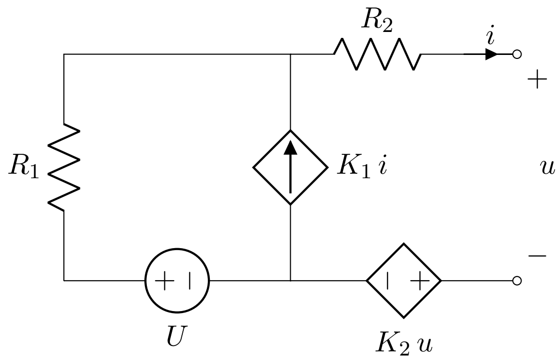
(It is *not* required that you solve or simplify the equations, although in fact it should be quite easy in this particular circuit.)



3) [4p]

a) [3.5p] Determine the Thevenin equivalent of this circuit at the terminals where quantities u and i are marked.

b) [0.5p] Comment on the "maximum power" available from the terminals of this circuit in the particular case where $U = 0$.



Section B. Transient Calculations

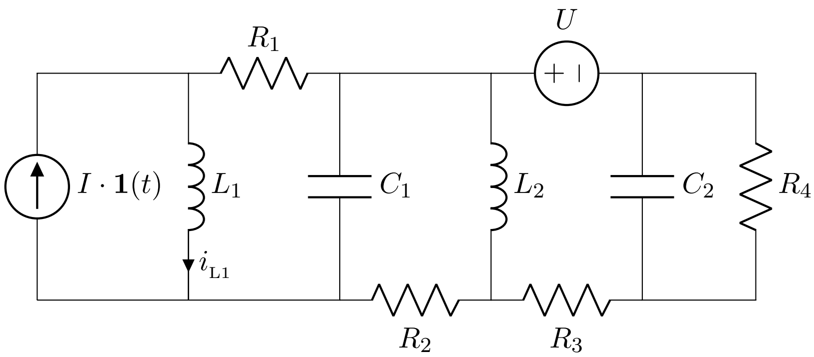
4) [5p] Find:

a) [1p] Energy in L_2 at $t = 0^-$

b) [1p] Power from I at $t = 0^+$

c) [2p] Power from C_2 at $t = 0^+$

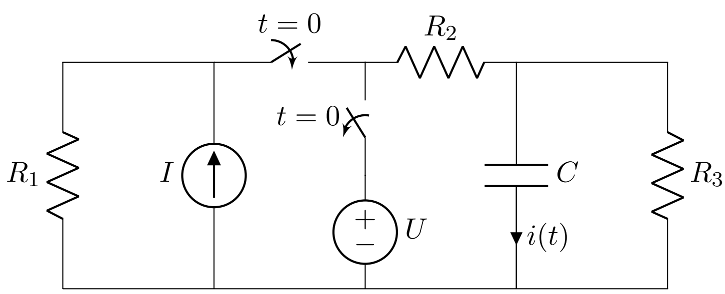
d) [1p] Current i_{L1} as $t \to \infty$



5) [5p]

a) [4p] Find the current $i(t)$ through the capacitor for $t > 0$.

b) [1p] Find the power into R_1 for $t > 0$.

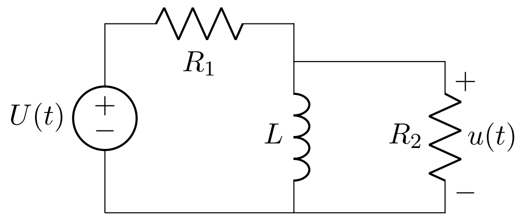


Section C. Alternating Current

6) [4p]

The source's voltage is $U(t) = \hat{U}\sin(\omega t)$.

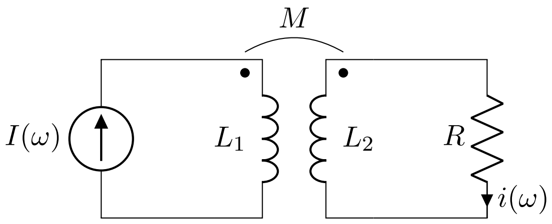
Determine $u(t)$.



7) [4p]

a) [2p] Determine this circuit's network function

$$H(\omega) = \frac{i(\omega)}{I(\omega)}.$$



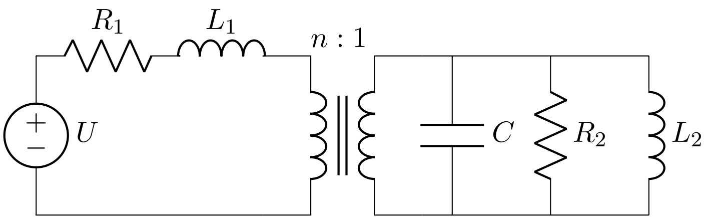
b) [1p] Show that the above $H(\omega)$ can be written as

$$H(\omega) = \frac{\mathrm{j}\omega/\omega_1}{(1+\mathrm{j}\omega/\omega_2)}.$$

c) [1p] Sketch a Bode amplitude plot of the function $H(\omega)$ from part 'b'. Assume $\omega_1 \ll \omega_2$. Mark significant points and gradients.

8) [4p]

The source has angular frequency ω.
Component value C can be chosen.
Other component values are fixed.



a) [3p] What value of C will fully power-factor compensate the load supplied from the right-hand side of the transformer, i.e. the total load consisting of C, R_2 and L_2? Express this value of C in terms of the other components' values.

b) [1p] What value of C will fully power-factor compensate the entire circuit supplied by source U? *In this part it is sufficient to show an equation that could be solved, without having to solve/rearrange to a neat expression for C.*

9) [6p]

Consider a balanced three-phase system according to the description below:

> A three-phase voltage source provides line-voltage magnitude U at angular frequency ω.
> A three-phase load is formed from three resistors each of value R, in star (Y) connection.
> The source and load are *not* directly connected. They are connected through three single-phase transformers that together form a three-phase transformer. Each of the transformers has N_1 turns on its primary winding and N_2 on its secondary. The primaries are connected with each other in star, and are supplied from the source. The secondaries are connected to each other in delta, and supply the load.

Advice! Think calmly, and draw carefully. Use symmetry where you can. Write what you know. Simplify.

a) [1p] What is the voltage magnitude across each load resistor?

b) [2p] What complex power is supplied by the source?

c) [2p] What is the magnitude of the current in the transformer secondaries?

d) [1p] One of the transformers has a fault where its secondary becomes fractured (open circuit). This is no longer a balanced three-phase circuit. What complex power does the source now supply? Think carefully!

The End. *Please don't waste remaining time ... check your solutions!*

Översättningar:

Hjälpmedel: Upp till tre A4-ark (båda sidor) med studentens egna anteckningar på valfritt sätt: handskrivet eller datorutskrift; text, diagram, bild; stor eller liten textstorlek, o.s.v. Dessa måste inte lämnas in med skrivningarna.

Om inte annan information anges i ett tal ska: komponenter antas vara ideala; angivna värden av komponenter (t.ex. R för ett motstånd, U för en spänningskälla, K för en beroende källa) antas vara *kända* storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller spänningskälla) antas vara *okända* storheter. Lösningar ska uttryckas i kända storheter och förenklas.

Var tydlig med diagram och definitioner av variabler. Du får skriva på svenska eller engelska, men vi rekommenderar att diagram och ekvationer används i stället i de flesta fall.

1. [4p] Bestäm följande:
a) [1p] effekten från källan U_2
b) [1p] potentialen v
c) [1p] effekten in till motstånd R_3
d) [1p] effekten från källan I_1

2. [4p] Skriv ekvationer som skulle *kunna* lösas, utan vidare information, för att bestämma de fem potentialerna $v_{1,2,3,4,5}$ som funktioner av kretsens komponentvärden. Det rekommenderas inte att du försöker lösa dem!

3. [4p]
a) [3.5p] Bestäm kretsens Theveninekvivalent med avseende på polera där u är markerad.
b) [0.5p] Kommentera om maximaleffekten som kan levereras från polerna i fallet $U = 0$.

4. [5p] Bestäm:
a) [1p] Energin lagrad i L_2 vid $t = 0^-$
b) [1p] Effekten levererad från I vid $t = 0^+$
c) [2p] Effekten levererad från C_2 vid $t = 0^+$
d) [1p] Strömmen i_{L1} vid $t \to \infty$

5. [5p]
a) [4p] Bestäm strömmen $i(t)$ genom kondensatorn, för $t > 0$.
b) [1p] Bestäm effekten till R_1, för $t > 0$.

6. [4p] Källans spänning är $U(t) = \hat{U}\sin(\omega t)$. Bestäm $u(t)$.

7. [4p]
a) [2p] Bestäm kretsens nätverksfunktion, i/I.
b) [1p] Visa att funktionen från deltal 'a' kan skrivas $\mathrm{j}\omega/\omega_1/(1 + \mathrm{j}\omega/\omega_2)$.
c) [1p] Skissa ett Bodeamplituddiagram av $H(\omega)$ från deltal 'b'. Antag $\omega_1 \ll \omega_2$. Markera viktiga punkter och lutningar.

8. [4p] Källan har vinkelfrekvens ω. Komponentvärdet C kan väljas, men andra komponentvärden är fasta.
a) [3p] Bestäm värdet av C som fullt effektfaktorkompenserar effekten levererad genom transformatorn.
b) [1p] Bestäm värdet av C som fullt effektfaktorkompenserar effekten levererad från källan U. Du *måste inte lösa* hela vägen: den räcker med ekvation som går att lösa.

9. [6p] Betrakta följande balanserade trefassytemet: En trefas spänningskälla levererar huvudspänning U vid vinkelfrekvens ω. Tre enfasiga transformatorer som har N_1 varv på sina primäralindningar och N_2 varv på sina sekundäralindningar kopplas för att skapa en trefastransformator. Primäralindningarna stjärnkopplas, och matas från källan. Sekundäralindningarna deltakopplas och matar en last som består av tre mostånd R i stjärnkoppling.
a) [1p] Vad är det för spänningsmagnitud över varje fas i lasten (varje motstånd R).
b) [2p] Bestäm den komplexeffekt som källan levererar.
c) [2p] Bestäm magnituden av strömmen i de sekundäralindningarna av transformatorna.
d) [1p] Obalanserat fall: sekundärlindningen av *en* av de tre transformatorna blir en öppenkrets, för att ett brott har skett i anslutningen. Bestäm den komplexeffekt som källan nu levererar till lasten. Tänk försiktigt.

Solutions (EI1120 TEN1 VT20, 2020-06-05)

Q1.

a) By KCL at the node marked v, the current out of the +-terminal of source U_1 is $I_1 + I_2$.
The product of the source's voltage and this current is then the power delivered from the source.

$$P_{\mathrm{U2}} = U_2 \left(I_1 + I_2\right).$$

b) The current up through R_2 is $I_1 + I_2$, which can be seen from KCL as in 'a'.
This gives a voltage $\left(I_1 + I_2\right) R_2$ on the bottom relative to the top part of that resistor.
Starting from zero potential, and summing the voltages as we pass through R_2 and U_2,

$$v = -\left(I_1 + I_2\right) R_2 + U_2.$$

c) R_3 and R_4 are connected in parallel. Although they're not drawn as parallel-lying components, you can check that they connect between the same pair of nodes. R_1 is then in series with these.

The branch formed from these three therefore has resistance $R_{123} = R_1 + \frac{R_3 R_4}{R_3 + R_4}$. That branch is in parallel with source U_1, so the voltage across it is fixed to U_1.

We can find the current through R_3 by finding the current that passes through R_1, which is the current through the whole branch, then using current division between R_3 and R_4,

$$i_{\mathrm{R3}} = \frac{R_4}{R_3 + R_4} \cdot \frac{U_1}{R_1 + \frac{R_3 R_4}{R_3 + R_4}} = \frac{U_1 R_4}{R_1 R_3 + R_1 R_4 + R_3 R_4}.$$

Or we can find the voltage across R_3 by voltage division of U_1 between R_1 and the parallel R_3 and R_4,

$$u_{\mathrm{R3}} = \frac{\frac{R_3 R_4}{R_3 + R_4} U_1}{R_1 + \frac{R_3 R_4}{R_3 + R_4}} = \frac{U_1 R_3 R_4}{R_1 R_3 + R_1 R_4 + R_3 R_4}.$$

One of the relations for a resistor's power, $P = i^2 R$ or $P = u^2/R$, can then be used to find P_{R3},

$$P_{\mathrm{R3}} = \frac{R_3 U_1^2 R_4^2}{\left(R_1 R_3 + R_1 R_4 + R_3 R_4\right)^2}.$$

d) The power from this source is found by the product of its value (I) and the voltage across it. The voltage should be marked in the direction with + at the terminal where the current I comes out: else a negation is needed in order to find the power out from the source.

The outer loop of the circuit is a good path for KVL to determine the voltage across source I. It contains just the unknown voltage across the current source, the two known values of voltage sources, and one resistor R_2. The current through R_2 is already known from part of the solution of 'b'.

$$u_{\mathrm{I1}} = U_1 + \left(I_1 + I_2\right) R_2 - U_2.$$

The sought power is therefore

$$P_{\mathrm{I1}} = I_1 \Big(U_1 - U_2 + \left(I_1 + I_2\right) R_2 \Big).$$

Q2.

Extended nodal analysis.

The plain, not-forced-to-think-very-hard way.
Simple rules to follow for writing the equations, but not so nice to solve!

Start with KCL at all five nodes other than the reference.
KCL can be written directly for nodes 1 and 5, based on currents defined by a current source or open-circuit, or currents in resistors that can be expressed in terms of the potentials and resistance. KCL at nodes 2, 3 or 4 meets the problem that there are unknown currents in voltage sources.

The circuit contains two obvious voltage sources: the independent U, and the dependent $K\,u_x$. It also has an opamp, whose output can be considered as one side of a voltage source whose other side connects to the reference node. The currents in these are not known: we *define* them as new unknowns, which we can call i_α into the +-terminal of U, then i_β into the +-terminal of Ku_x, and i_γ into the opamp output.

$$\text{KCL(1)}: \quad 0 = I + \frac{v_1}{R_1} \tag{1}$$
$$\text{KCL(2)}: \quad 0 = -I + \frac{v_2}{R_2} + i_\beta \tag{2}$$
$$\text{KCL(3)}: \quad 0 = -i_\beta + \frac{v_3}{R_3} + i_\alpha \tag{3}$$
$$\text{KCL(4)}: \quad 0 = -i_\alpha + \frac{v_4}{R_4} + i_\gamma + 0 \tag{4}$$
$$\text{KCL(5)}: \quad 0 = \frac{v_5}{R_5} + 0 \tag{5}$$

The above are 5 equations, in 8 unknowns. The unknown currents in the three voltage sources are the reason for this difference. The same sources provide the further necessary equations, as they set relations between pairs of node-potentials. The opamp is a special case: its negative feedback and high gain result in its *input* potentials being equal, which may seem surprising given that the *output* current is the unknown . . . but it works.

$$v_3 - v_4 = U \tag{6}$$
$$v_2 - v_3 = K\,u_x \tag{7}$$
$$v_4 - v_5 = 0 \tag{8}$$

The first of these introduced a new unknown, u_x, as the controlling variable of the dependent source. There are now therefore 9 unknowns and 8 equations. The controlling variable has to be defined as an equation, in order to convey the same information as the diagram tells us about them; otherwise the equations don't provide enough information to solve the shown circuit, since the result of course depends on how the controlling variable is defined.

$$u_x = v_4 - 0. \tag{9}$$

The above 9 equations in 9 unknowns should be able to be solved. We can be confident of this because of having followed a particular procedure: beware of writing linearly-dependent equations if not being systematic!

Step-by-step solution.

This circuit is unusually easy to solve for the potentials *without* using a systematic method or simultaneous equations. As with many puzzles, finding the right starting-point makes it much simpler.

No current can pass through R_5, as this resistor connects only to an opamp input. With zero current through it, Ohm's law says this resistor has zero voltage. So $v_5 = 0$.

The ideal opamp with negative feedback must have its inputs at the same potential, so $v_4 = 0$.

The voltage source U forces $v_3 = v_4 + U$, so $v_3 = U$.

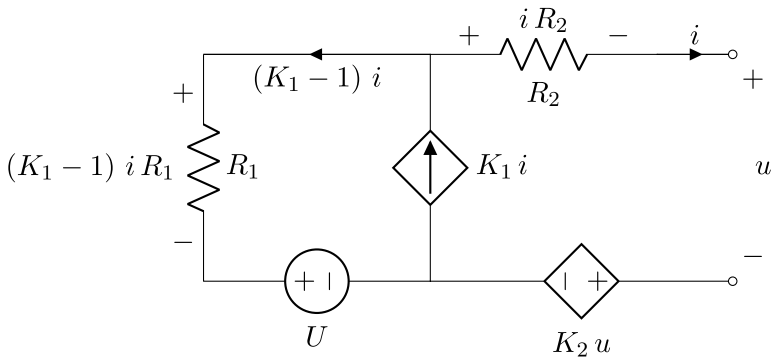
Similarly, the dependent voltage source $K\,u_x$ determines that $v_2 - v_3 = u_x$. The definition of u_x across R_4 shows that $u_x = -v_4$. With the value of $v_4 = 0$, we this gives $u_x = 0$. Putting this into the equation for the dependent source, $v_2 = U$.

Finally, the current through R_1 is determined by the source I, and one side of R_1 connects to the reference node (0). Taking into account the directions, $v_1 = -IR_1$.

All the five unknown potentials have been determined in the above. The same could have been done quite easily from the equations found in the nodal analysis method, taking equations (5), (8), (6), (9), (7) and (1).

Q3.

The voltages across the two resistors can be determined in terms of the marked current i.
In R_2 the current is i.
In R_1 it is the difference between i and $K_1 i$, by KCL above the current-source.



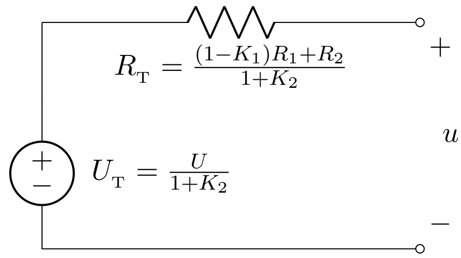
Using these voltages, KVL can be written around the outer loop,

$$-K_2\,u + U + R_1\,(K_1 i - i) - R_2\,i - u = 0,$$

which directly gives a relation between the two terminal-quantities (u & i) in terms of the known component values:

$$u = \frac{U}{1+K_2} - \frac{(1-K_1)\,R_1 + R_2}{1+K_2}\,i.$$

a) The Thevenin parameters are as shown on the right. They can be found by comparison of the above with the equation of a Thevenin source, $u = U_{\mathrm{T}} - R_{\mathrm{T}}\,i$.



Alternatively, short-circuit and open-circuit conditions can be studied to find short-circuit current i_{sc} and open-circuit voltage $u_{\mathrm{oc}} = U_{\mathrm{T}}$, and their ratio $R_{\mathrm{T}} = u_{\mathrm{oc}}/i_{\mathrm{sc}}$.

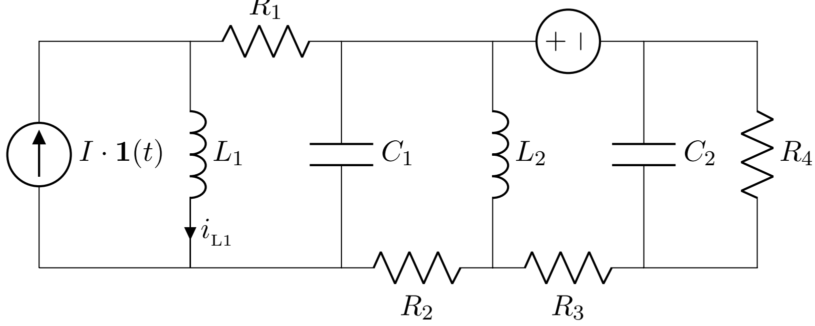
In short-circuit $u = 0$, so the dependent voltage-source is fixed to zero, i.e. a short circuit. Drawing the rest of the circuit, with the terminals shorted, and writing KVL around the outer loop similarly to the above (but without u or $K_2\,u$), we get $U + R_1(K_1-1)i - R_2 i = 0$, from which $i_{\mathrm{sc}} = \frac{U}{(1-K_1)R_1+R_2}$.

In open-circuit $i = 0$, so the dependent current source is fixed to zero, i.e. an open circuit, and can be omitted. Just one loop remains, and its current is i, so it is zero. By Ohm's law the resistors then have zero voltage. By KVL around the loop, $-K_2 u + U + 0 + 0 - u = 0$, from which $u_{\mathrm{oc}} = \frac{U}{1+K_2}$.

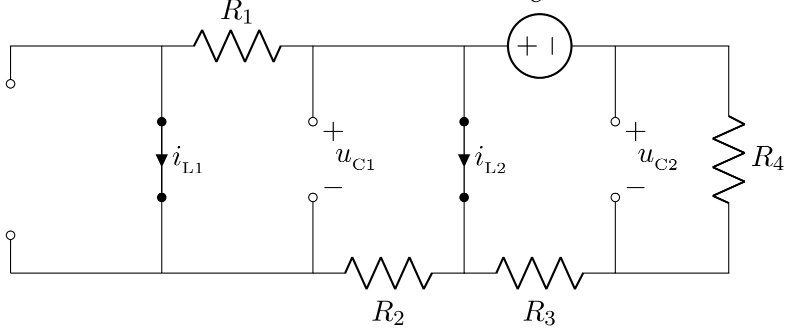
b) If $U = 0$ then $U_{\mathrm{T}} = 0$. The circuit then behaves like just a resistor R_{T}, and if this has a positive value the circuit can only absorb, not supply, power. Depending on the parameters K_1 and K_2, R_{T} could be negative: in this case the circuit will supply power to the current that passes through it, but there's no maximum to the power, as increased current will always mean increased power.

Q4.

The original circuit, describing all times, is the following:



At $t = 0^-$, the step function makes the current-source have zero value, so it can be represented by an open circuit. Equilibrium implies that capacitors have no current (open) and inductors have no voltage (short).



Just the energy stored in inductor L_2 is requested, for this time $t = 0^-$.

Because C_1 behaves as an open circuit, R_3 and R_4 are in series. From KVL around $\{\ U,\ L_2,\ R_3,\ R_4\ \}$, the voltage across this series pair is U, so the current through R_3 is $\frac{U}{R_3+R_4}$.

There is no current in the components to the left of L_2, as all that part is connected across the zero voltage of L_2 and has no active source. So no current flows in R_2. If a more rigorous demonstration is wanted, the zero current in R_2 can be shown by KVL around $\{\ R_2,\ L_1,\ R_1,\ L_2\ \}$, after using KCL to show that the currents in R_1 and R_2 are the same.

By KCL between R_2 and R_3, the current in L_2 is therefore the same as the current through R_3. Putting this into the formula for energy in an inductor, we get the solution to 'a':

a) $W_{L2}(0^-) = \frac{1}{2} L_2 \left(\frac{U}{R_3+R_4} \right)^2.$

Before moving onto $t = 0^+$ and the next sub-question, we should analyse the state at $t = 0^-$ further in order to find the continuous quantities in the capacitors and inductors. (We probably aren't sure yet whether all of these quantities will actually be needed for solving the later questions. But the circuit at $t = 0^-$ looks as if it's quite easy to find all four continuous quantities, so we'll do that anyway.)

From the above analysis, there is no current in R_1 or R_2, so:

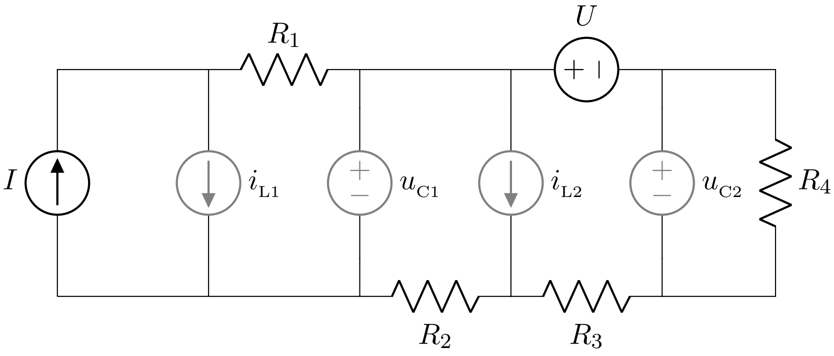
By KCL, $i_{L1} = 0$.
By KVL, $u_{C1} = 0$.

The current already determined in R_3 and R_4 leads to:

As found earlier by KCL, $i_{L2} = \frac{U}{R_3+R_4}$.
By voltage division, $u_{C2} = \frac{-UR_4}{R_3+R_4}$.

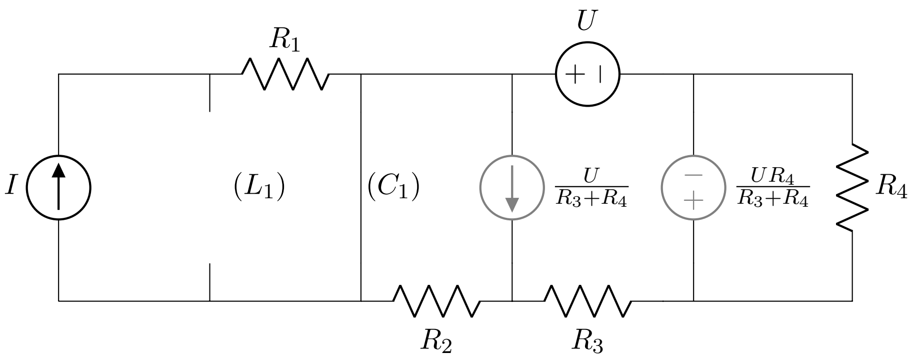
At $t = 0^+$ the step function gives the current-source a value I. Continuity implies that the capacitors and inductors maintain, respectively, the voltage and currents that they had previously. (These values will change over time, but are not changed in the negligible time between 'just before' and 'just after' the step function.)

We can represent these still-unchanged values by replacing capacitors with voltage sources and inductors with current sources, with values matching the continuous variables found at $t = 0^-$.



When putting in the values, we note that $i_{L1} = 0$ describes a zeroed current-source, which can more simply be written as an open circuit. Similarly, the zeroed voltage-source $u_{C1} = 0$ can be written as a short circuit. The negative sign in the expression for u_{C2} can be avoided by swapping the direction of that source.

The result is a simpler diagram for $t = 0^+$, that is clearer to think about.



The power delivered from source I depends on the voltage across this source. KCL shows that the full current I passes through R_1. Taking KVL around the loop of { I, R_1 (C_1) }, this shows that the voltage across the current source is IR_1, with its reference direction (+-side) being the upper terminal of source I. Multiplying this with the source current gives the power delivered from the source,

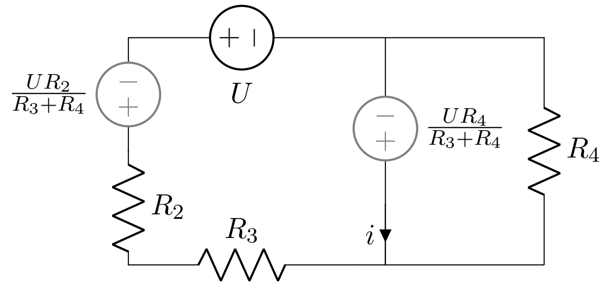
b) $P_I(0^+) = I^2R_1$.

Similarly, the power delivered from C_2 can be found by first finding this component's current.

Define a current i downwards, out of the +-terminal of the source that represents C_2.

The source $U/(R_3+R_4)$ is in parallel with R_2, so these can be source-transformed to a Thevenin source.

The circuit on the right can then be analysed to find i.



Writing KCL at the node below the capacitor C_2 (represented by the source $UR_4/(R_3+R_4)$), the current i is expressed as the sum of the currents passing to the left and to the right. Each of these loops is a series connection of voltage sources and resistors, in which KVL can be used to find the currents.

$$i = \frac{\frac{UR_4}{R_3+R_4} - U - \frac{UR_2}{R_3+R_4}}{R_2+R_3} + \frac{\frac{UR_4}{R_3+R_4}}{R_4}.$$

The components I and R_1 were neglected in all of this analysis, as they connect by only on one node to the part of the circuit that we are studying, so they cannot affect the currents or voltages beyond that node. If you aren't happy with the above justification for ignoring part of the circuit, you could use nodal analysis on the right part of the circuit without doing a source transformation. For example, take the potential above or below R_4 as the reference: then the only unknown potential that is needed in order to find i is at the node between R_2 and R_3. KCL at this node doesn't require I or R_1, as the potential at the left of R_2 is the same as the known potential at the left of U.

Taking the above expression for i and putting it over a common denominator,

$$i = U\,\frac{R_4^2 - R_3R_4 - R_4^2 - R_2R_4 + R_2R_4 + R_3R_4}{(R_3+R_4)\,(R_2+R_3)\,R_4} = 0.$$

This result may come as a surprise. It means that there is no power out of C_2, because C_2 has no current at $t=0^+$. (There was also no current in C_2 at $t=0^-$, but current is not the continuous variable of a capacitor, so we can't in general assume it to be unchanged.)

In fact, we didn't need to do all this algebra to see that i would still be zero: there was a shortcut. But we did the long way anyway, to show that it's anyway possible to find the solution by hard work and rule-following.

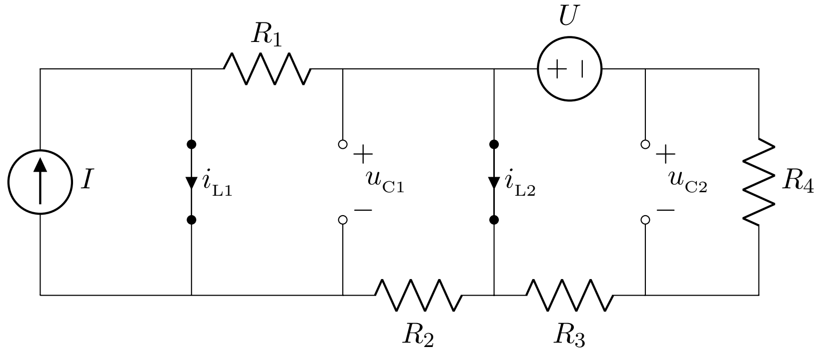
The shortcut comes from noticing that the only change that happens in the circuit is the current source at the left getting a different value at $t=0$. Between that source and the region we're interested in with C_2, there is a capacitor C_1 connected in parallel. That capacitor, by continuity, prevents a change in its voltage between $t=0^-$ and $t=0^+$. By this behaviour, like a voltage source in parallel with the part we're studying, C_1 prevents the change at I from being 'seen' by the components to its right, immediately after the change. So we could have predicted that the capacitor's current would have been unchanged from its value at $t=0^-$, even though it is *not* continuous variable.

Later, after $t=0^+$, the capacitor C_1 will have had time to change its voltage with the new current coming in through R_1. Then C_2 and other components can get different voltages and currents.

c) $P_{\mathrm{C2}}(0^+) = i\frac{UR_4}{R_3+R_4} = 0\frac{UR_4}{R_3+R_4} = 0.$

As $t\to\infty$, a new equilibrium is reached, differing from the case at $t=0^-$ in that the current source is now I instead of 0.

By KCL, the currrent down L_1 is the sum of I and the current out of the right terminal of R_1. The current in R_1 is zero. This can be seen from KVL around $\{\ R_1,\ L_2,\ R_2,\ L_1\ \}$, bearing in mind that by KCL the same current must flow around both of these resistors.

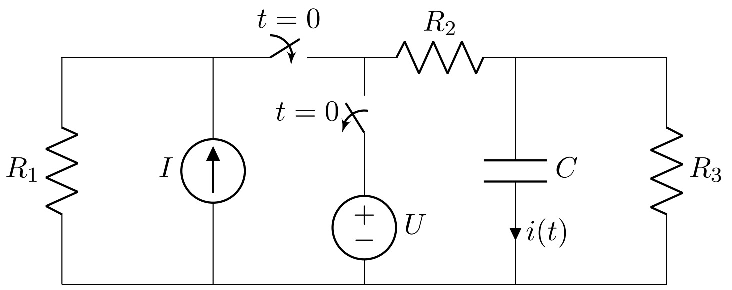


The solution is therefore that all the source current I, and nothing more, passes through L_1.

d) $i_{\mathrm{L1}}(\infty) = I.$

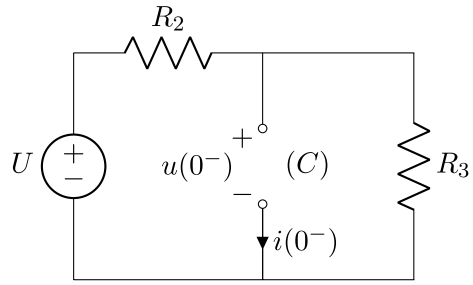
Q5.

The circuit for all times is this:



The equilibrium at $t=0^-$ allows us to find the initial condition of the capacitor's continuous variable, which is needed in the solution of what happens for $t>0$. The simplified circuit for $t=0^-$ is shown on the right.

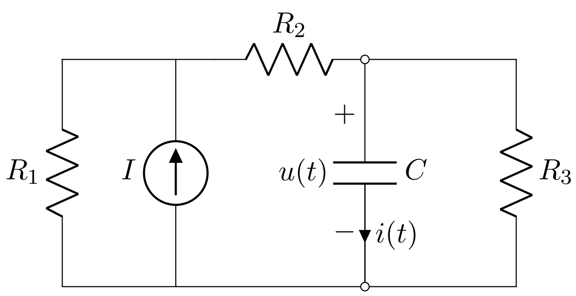
At $t=0^-$, the switch to the voltage source is closed, and the other switch is open, preventing components I and R_1 from affecting the capacitor. The capacitor in equilibrium is an open circuit.

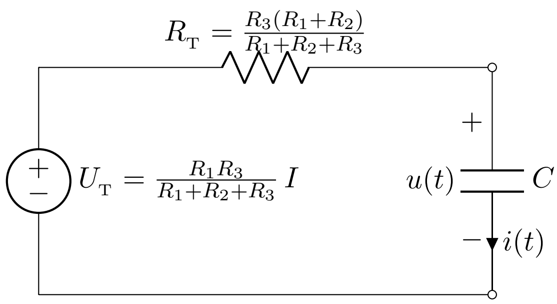


Voltage division finds the voltage u across the capacitor in this equilibrium. From the above circuit, and continuity,

$$u(0^+) = u(0^-) = \frac{R_3}{R_2+R_3}\,U.$$

After the switches have operated, the voltage source U is disconnected, but the Norton-source of { I & R_1 } is connected to the rest of the circuit. This is shown at the left, below. At the right, all of the circuit except the capacitor has been replaced by its Thevenin equivalent.





Notes on deriving the Thevenin equivalent: The capacitor is not included, so it can be treated as removed (open). R_2 and R_3 are then in series, and this branch is in parallel with R_1 and with I. Current division finds the proportion of I passing to the right, which is multiplied by R_3 to find the open-circuit voltage between the nodes where the capacitor will connect. The Thevenin resistance is found by setting source I to zero, so it's open circuit. Then R_3 is in parallel with R_1+R_2, when seen from the nodes where the capacitor will connect.

Using the quick method to find the ODE in a circuit with just one C or L,

$$u(0^+) = \frac{R_3}{R_2+R_3}U, \qquad u(\infty) = U_{\mathrm{T}} = \frac{R_1R_3}{R_1+R_2+R_3}I, \qquad \tau = CR_{\mathrm{T}} = C\,\frac{R_3\,(R_1+R_2)}{R_1+R_2+R_3}.$$

which gives $u(t)$ for $t>0$ as

$$u(t) = u(\infty) + \big(u(0^+) - u(\infty)\big)\,\mathrm{e}^{-t/\tau} = \frac{R_1R_3I}{R_1+R_2+R_3} + \left(\frac{R_3U}{R_2+R_3} - \frac{R_1R_3I}{R_1+R_2+R_3}\right)\,\mathrm{e}^{-t\frac{R_1+R_2+R_3}{C\,R_3(R_1+R_2)}}.$$

a) The first task was to find the current $i(t)$. We initially found the voltage instead, as a capacitor's voltage is its continuous variable, so its value just after the change is the same as in the equilibrium just before that.

Now we use the equation of a capacitor, and insert the above expression for $u(t)$,

$$i(t) = C\frac{\mathrm{d}u(t)}{\mathrm{d}t} = C\,\frac{-(R_1+R_2+R_3)}{C\,R_3(R_1+R_2)}\left(\frac{R_3U}{R_2+R_3} - \frac{R_1R_3I}{R_1+R_2+R_3}\right)\mathrm{e}^{-t\frac{R_1+R_2+R_3}{C\,R_3(R_1+R_2)}}.$$

Trying to simplify a bit,

$$i(t) = \frac{R_1+R_2+R_3}{R_1+R_2}\left(\frac{R_1I}{R_1+R_2+R_3} - \frac{U}{R_2+R_3}\right)\mathrm{e}^{-t\frac{R_1+R_2+R_3}{C\,R_3(R_1+R_2)}},$$

$$i(t) = \left(\frac{R_1I - \frac{R_1+R_2+R_3}{R_2+R_3}U}{R_1+R_2}\right)\mathrm{e}^{-t\frac{R_1+R_2+R_3}{C\,R_3(R_1+R_2)}} \qquad (t>0).$$

Any form of simplification that cancels the C should be accepted, as there is not a really obviously much neater way to express this result.

b) Now we should find the power into R_1 for $t>0$. It looks as if finding the current in this resistor is a good way to start.

If i_1 is the current down R_1, KCL gives

$$i_1(t) = I - i(t) - \frac{u(t)}{R_3}.$$

The above can be seen as KCL taken on currents into or out of the entire top part of the circuit. Or it can be seen as a KCL on the top right node to find the current through R_2, followed by KCL on the top left.

(An alternative to finding the current in R_1 is to find the voltage across R_1, based on the current in R_2 found from KCL in the top right node, and the known voltage $u(t)$. But we'll go with the current, as it doesn't require the expression for $u(t)$ to be written out twice.)

Putting in the earlier calculated values of $u(t)$ and $i(t)$,

$$i_1(t) = I - \frac{R_1+R_2+R_3}{R_1+R_2}\left(\frac{R_1I}{R_1+R_2+R_3} - \frac{U}{R_2+R_3}\right)\mathrm{e}^{-t/\tau} - \frac{1}{R_3}\left(\frac{R_1R_3I}{R_1+R_2+R_3} + \left(\frac{R_3U}{R_2+R_3} - \frac{R_1R_3I}{R_1+R_2+R_3}\right)\mathrm{e}^{-t/\tau}\right).$$

Trying to improve this mess, we separate the constant parts and the exponentially decaying parts,

$$i_1(t) = \left(1 - \frac{R_1}{R_1+R_2+R_3}\right)I + \left(\left(\frac{R_1+R_2+R_3}{(R_1+R_2)\,(R_2+R_3)} - \frac{1}{R_2+R_3}\right)U + \left(\frac{R_1}{R_1+R_2+R_3} - \frac{R_1}{R_1+R_2}\right)I\right)\mathrm{e}^{-t/\tau},$$

then do some simplifications by combining fractions,

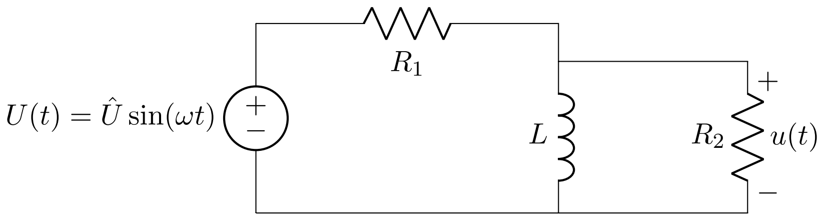
$$i_1(t) = \frac{R_2+R_3}{R_1+R_2+R_3}I + \left(\frac{R_3}{(R_1+R_2)\,(R_2+R_3)}U - \frac{R_1R_3}{(R_1+R_2)\,(R_1+R_2+R_3)}I\right)\mathrm{e}^{-t\frac{R_1+R_2+R_3}{C\,R_3(R_1+R_2)}}.$$

The solution is then

$$P_{\mathrm{R1}}(t>0) = i_1(t)^2R_1.$$

There's no obvious advantage to copying the full expression for i_1 into this, as that will just get even worse to understand, especially if we expand out the squared part.

Q6.



The aim is to determine $u(t)$, by ac analysis.

L and R_2 are in parallel, then in series with R_1 across a known voltage. We want to find the voltage across one of these series parts. This is a good case for voltage division.

The inductor's impedance is $Z = \mathrm{j}\omega L$.

We'll represent the source's voltage as a phasor $U(\omega) = \hat{U}\underline{/0}$. This uses sine-reference for angle, and peak-value for magnitude; the same choices must be made later, when transforming the result back to a time-function. We don't really need to write the $\underline{/0}$ part, but it helps remind that this is a phasor although it has zero phase.

Now we can do the same as we would in a dc circuit with a voltage source and three resistors:

$$u(\omega) = \frac{\dfrac{ZR_2}{Z+R_2}}{R_1 + \dfrac{ZR_2}{Z+R_2}}\,U(\omega) = \frac{ZR_2 \cdot U(\omega)}{R_1R_2 + Z\,(R_1+R_2)}.$$

Having simplified a bit with the simpler symbols above, we can put in the values,

$$u(\omega) = \frac{\mathrm{j}\omega L R_2\,\hat{U}\underline{/0}}{R_1R_2 + \mathrm{j}\omega L\,(R_1+R_2)} = \frac{\mathrm{j}\omega L/R_1 \cdot \hat{U}\underline{/0}}{1 + \mathrm{j}\omega L\frac{R_1+R_2}{R_1R_2}}.$$

The final form above, with a $1+\mathrm{j}\omega/\omega_0$ type of term, would be good for Bode plotting. It's not necessarily going to be better for our purposes here, but it looks about as good as any.

Now, in order to write the time function $u(t)$, we must find the magnitude and angle of $u(\omega)$ and put them into an equation based on the peak-magnitude sine-reference that we used to convert from $U(t)$ to $U(\omega)$,

$$u(t) = |u(\omega)|\,\sin\left(\omega t + \underline{/u(\omega)}\right).$$

The magnitude, based on either of the two expressions for $u(\omega)$ above, is

$$|u(\omega)| = \frac{\hat{U}\omega L/R_1}{\sqrt{1+\left(\omega L\frac{R_1+R_2}{R_1R_2}\right)^2}} \qquad \text{or} \qquad |u(\omega)| = \frac{\omega L R_2 \hat{U}}{\sqrt{(R_1R_2)^2 + (\omega L\,(R_1+R_2))^2}}.$$

The phase can be found by directly taking the phase of the top and bottom parts,

$$\underline{/u(\omega)} = \operatorname{atan}\frac{\omega L/R_1}{0} - \operatorname{atan}\frac{\omega L\frac{R_1+R_2}{R_1R_2}}{1} = \frac{\pi}{2} - \operatorname{atan}\omega L\frac{R_1+R_2}{R_1R_2},$$

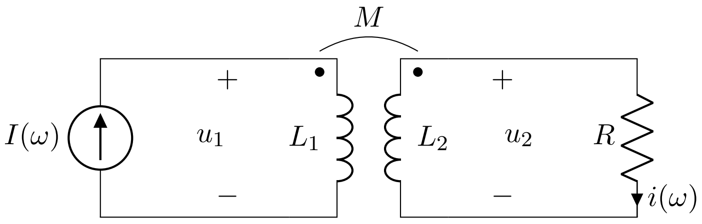
or by first dividing both by j to make the top part real and thereby avoid multiple atan terms,

$$\underline{/u(\omega)} = 0 - \operatorname{atan}\frac{-1}{\omega L\frac{R_1+R_2}{R_1R_2}} = \operatorname{atan}\frac{1}{\omega L\frac{R_1+R_2}{R_1R_2}} = \operatorname{atan}\frac{R_1R_2}{\omega L\,(R_1+R_2)}.$$

The final answer is then:

$$u(t) = \frac{\hat{U}\omega L/R_1}{\sqrt{1+\left(\omega L\frac{R_1+R_2}{R_1R_2}\right)^2}}\,\sin\left(\omega t + \operatorname{atan}\frac{R_1R_2}{\omega L(R_1+R_2)}\right).$$

Q7.



a) For this mutual-inductor pair we could write the usual two equations, for the voltages u_1 and u_2 at both sides, in terms of the two currents. For the inductor L_2, the equation would be

$$u_2 = \mathrm{j}\omega L_2(-i) + \mathrm{j}\omega M I,$$

where the negative sign on the current i is because of the relative direction of the marked voltage u_2, the dot on L_2, and the current i.

Then we could write further expressions for how these voltages and currents are related in the circuits that connect to the inductors. On the right, this relation is

$$u_2 = i\,R.$$

In this circuit it is not necessary to write equations for the left side of the circuit. This is because the current-source completely determines the current in the coil L_1, regardless of what happens at the other side of the circuit. Thus, the influence of the left side on the right side is a constant, independent of what's connected at the right.

(Notice that this would *not* be the case if there were for example a voltage source or Thevenin or Norton source on the left: in that case, the current in L_1 *would* depend on the voltage u_1, which would depend also on the current in L_2, which would depend on the current in L_1, leading to simultaneous equations.)

In order to solve for the marked i, we can take the above two equations, use one to eliminate u_2 in the other, and thereby get one equation for i in terms of known quantities,

$$i = \frac{\mathrm{j}\omega M I}{R + \mathrm{j}\omega L_2}, \qquad H(\omega) = \frac{i}{I} = \frac{\mathrm{j}\omega M}{R + \mathrm{j}\omega L_2}.$$

b) The above equation can be simply modified by dividing through by R, giving

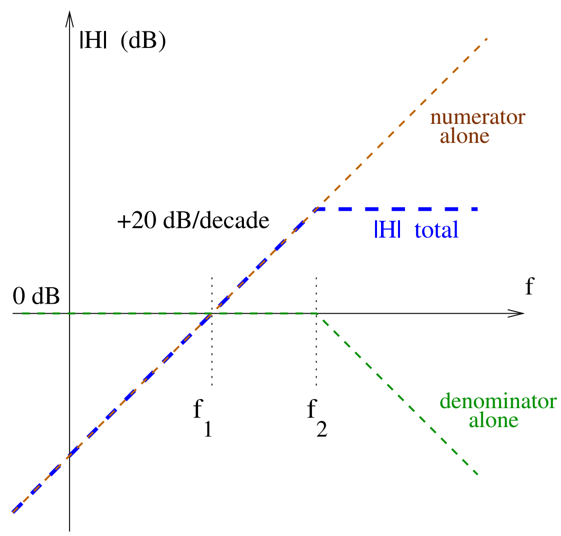
$$\frac{i}{I} = \frac{\mathrm{j}\omega M/R}{1 + \mathrm{j}\omega L_2/R} = \frac{\mathrm{j}\omega/\omega_1}{1 + \mathrm{j}\omega/\omega_2}, \quad \text{where} \quad \omega_1 = \frac{R}{M} \quad \text{and} \quad \omega_2 = \frac{R}{L_2}.$$

c) A Bode amplitude plot of the function from 'b' is shown on the right. The parts for numerator and denominator terms alone are not required, but are simply included to show a way of making the total function from these parts.

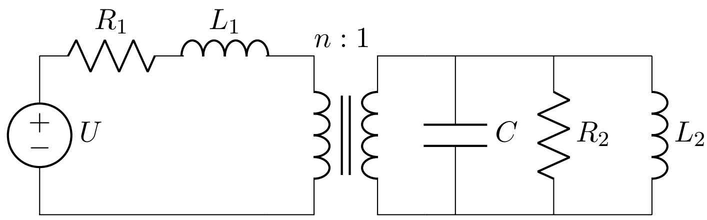
The $+20\,\mathrm{dB/decade}$ slope for $f < f_2$ should be marked. The zero slope at higher frequencies can be considered obvious (not required).

The $0\,\mathrm{dB}$ level and the crossing of $|H(\omega)|$ through this at frequency f_1 should both be clear.

The horizontal axis can be marked with a frequency f or angular frequency ω, with the assumption that e.g. f_1 here corresponds to ω_1 in the equation of 'b' by $\omega_1 = 2\pi f_1$.



Q8.



a) First, we aim to choose C to give perfect power-factor compensation of the total load supplied by the transformer.

In other words, no reactive power should flow between the right-hand side of the transformer and the set of three components to the right of it.

Or in another form, $\Im\{Z_{\mathrm{C}} \parallel Z_{\mathrm{R}} \parallel Z_{\mathrm{L2}}\} = 0$, where $\parallel$ denotes a parallel connection and $\Im\{\cdot\}$ the imaginary part of a complex number.

The requirement of a purely real total impedance of the three components implies also that the total *admittance* (reciprocal of impedance) is real. Admittance is probably easier to handle here, as the components are in parallel. The admittance of the three parallel components is

$$Y_{\parallel} \;=\; \frac{1}{R_2} + \mathrm{j}\omega C + \frac{1}{\mathrm{j}\omega L_2} \;=\; \frac{1}{R_2} + \mathrm{j}\left(\omega C - \frac{1}{\omega L_2}\right).$$

The imaginary part of this is zero if

$$C = \frac{1}{\omega^2 L_2}.$$

b) Now the power-factor compensation should make the source see a unity power factor (no reactive power). This is different from 'a' in that the reactive power in the inductor L_1 must also be compensated; or in other words, the total impedance of everything seen by source U must be purely real.

The solution gets a bit messy because the compensation is done with a component C that lies on the other side of L_1 from the source: if we change C to try to compensate for the reactive power in L_1, our action changes the current through L_1, which changes the reactive power in L_1 and thus the needed value of C. It would be simpler if C were in parallel with the source. Instead we end up with C in several places in the equations.

Let's write the total impedance seen by the source U. First take the reciprocal of the $Y_{\parallel}$ found in 'a', then refer it to the primary side of the transformer by scaling by n^2, then add the series impedances from the primary side:

$$Z_{\text{total}} \;=\; R_1 + \mathrm{j}\omega L_1 + n^2\left(\frac{1}{\frac{1}{R_2} + \mathrm{j}\omega C + \frac{1}{\mathrm{j}\omega L_2}}\right).$$

In view of the question's instructions, it's ok to leave it as above. It won't look much neater with rearrangement. The C appears only in one place. The most practical way to solve it, outside an exam, would be to plot the imaginary part of the expression as a function of C for given values of the other parameters, and check what C gives a zero imaginary part.

If one wants to go further, the following is an example.

$$Z_{\text{total}} \;=\; R_1 + \mathrm{j}\omega L_1 + \left(\frac{n^2\,\mathrm{j}\omega L_2 R_2}{R_2\left(1-\omega^2 L_2 C\right) + \mathrm{j}\omega L_2}\right) \;=\; R_1 + \mathrm{j}\omega L_1 + \left(\frac{n^2 \omega L_2 R_2}{\omega L_2 + \mathrm{j}R_2\left(\omega^2 L_2 C - 1\right)}\right)$$

Now get rid of imaginary parts on the bottom of the rightmost term, by multiplying top and bottom by the complex conjugate,

$$Z_{\text{total}} \;=\; R_1 + \mathrm{j}\omega L_1 + \left(\frac{n^2\omega^2 L_2^2 R_2 - \mathrm{j}n^2\omega L_2 R_2^2\left(\omega^2 L_2 C - 1\right)}{\omega^2 L_2^2 + R_2^2\left(\omega^2 L_2 C - 1\right)^2}\right)$$

The required condition is a zero imaginary part. This means that $j\omega L_1$ must be cancelled by the imaginary part of the term on the right.

Expressing just these imaginary parts as equal and opposite, we get

$$\omega L_1 \left(\omega^2 L_2^2 + R_2^2 \left(\omega^2 L_2 C - 1\right)^2\right) = n^2 \omega L_2 R_2^2 \left(\omega^2 L_2 C - 1\right)$$

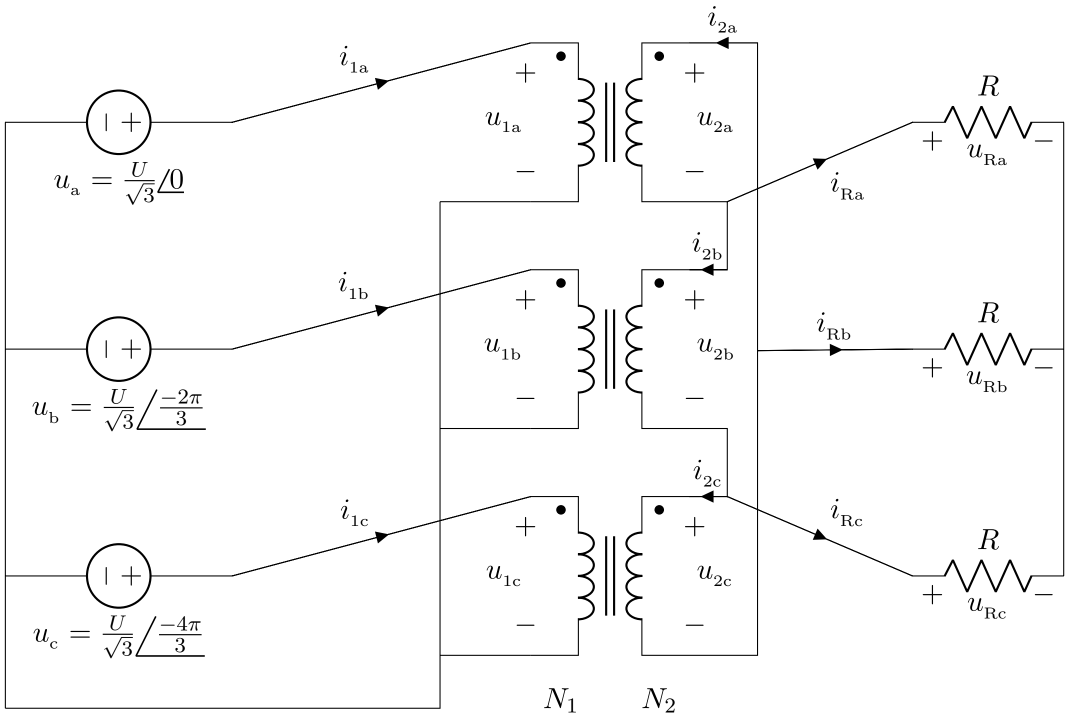
After more fooling around, this can become a quadratic in C,

$$\omega^2 L_2 C^2 - \left(2 + n^2 L_2 / L_1\right) C + \left(\frac{L_2}{R_2^2} + \frac{n^2}{\omega^2 L_1} + \frac{1}{\omega^2 L_2}\right) = 0$$

to be solved with the quadratic formula.

Q9.

For all except the final part of this question, the described circuit can be modelled with the following diagram. Note that connections are only at triple-points: cross-overs of straight lines are not connected. The transformers are identical. The angles shown for the sources are just examples that were not specified in the question: any arbitrary angle could be added to all three and they would still be a balanced three-phase set.



The situation in the first three questions is balanced three-phase. Magnitudes therefore come in threes: they will be the same in each of the three source-phases, resistors, primary windings etc. Angles between the three phases will be $\pm 120°$.

continued ...

a) Voltage magnitude across each load-resistor R.

As it's a balanced three-phase situation, we can use standard relations between phase- and line-quantities. It's made simpler by the fact that we are only asked about magnitudes.

The transformer primaries are star-connected to a line-voltage of U, so each gets voltage magnitude $\frac{U}{\sqrt{3}}$.

Each transformer secondary has voltage magnitude $\frac{UN_2}{\sqrt{3}N_1}$, according to the transformer voltage equation.

The transformer secondaries are delta-connected, so they produce a line-voltage magnitude equal to their individual magnitudes. The line voltage applied to the resistive load is therefore $\frac{UN_2}{\sqrt{3}N_1}$.

The load resistors are star-connected, so each gets $\frac{1}{\sqrt{3}}$ of the line-voltage: $\frac{1}{\sqrt{3}} \cdot \frac{UN_2}{\sqrt{3}N_1}$. Thus,

$$\left|u_{\mathrm{R\{a,b,c\}}}\right| = \frac{UN_2}{3N_1}.$$

b) Complex power supplied by the source.

The transformers are assumed ideal, so they consume or produce no active or reactive power.

The three-phase source therefore delivers the same complex power as goes into the resistors. Each resistor has a voltage as found in 'a'.

$$S_{\mathrm{source}} = 3\frac{|u_{\mathrm{R}}|^2}{R} = 3\frac{\left(\frac{UN_2}{3N_1}\right)^2}{R} = \frac{U^2N_2^2}{3RN_1^2}.$$

c) Current magnitude in transformer secondaries.

One method: each load resistor's current magnitude is $|i_{\mathrm{R}}| = \frac{UN_2}{3N_1R}$, based on the solution to 'a'.

The resistors are star-connected, so this is also the line-current magnitude from the transformer to the resistors.

The transformer secondaries are delta-connected, so each has a current magnitude of $\frac{1}{\sqrt{3}}$ of the line current:

$$|i_2| = \frac{1}{\sqrt{3}}|i_{\mathrm{R}}| = \frac{UN_2}{3\sqrt{3}N_1R}.$$

d) Complex power supplied by the source, when one transformer secondary is open-circuit.

Due to the initial symmetry, it doesn't matter which transformer's secondary becomes disconnected. If we choose a different one, the situation will be the same except for the different phase angles. The sum of power on all three phases will therefore not be affected by the choice of which transformer is disconnected.

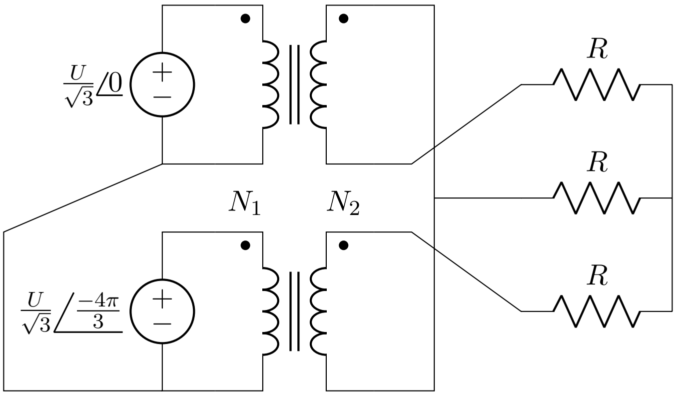
The principle from 'b' can still be used: whatever complex power leaves the source is the same as what goes in to the resistors.

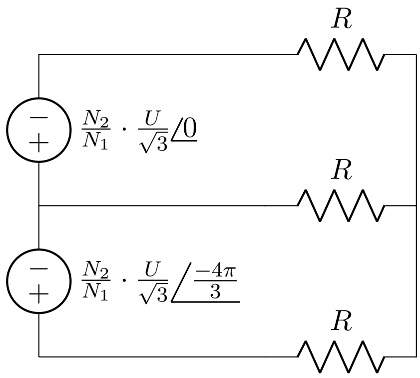
If one winding in an ideal transformer is open-circuit, the other must behave also as an open circuit, since $I_1 = I_2N_2/N_1$. The transformer with an open-circuit secondary can therefore be removed from the circuit, since it will behave as an open circuit at both sides.

Now there are two possibilities: were the source and transformer-primaries in a 3-wire or 4-wire (with neutral) connection? In the balanced conditions this would make no difference. But now it does. This was not specified in the question, because: i) identifying the issue was one possible way of getting credit in this question, ii) one or other choice could be made, and will get full credit for a correct answer for the specified choice.

1. *If there is a neutral from the source to the primaries:*

Let's choose the middle transformer (see earlier diagram) as the one with an open-circuit. After removing this transformer, the middle phase of the source is left unconnected, so can also be removed as it no longer does anything in the circuit. This leads to the diagram on the left. On the right, the diagram has been simplified by replacing the transformer secondaries by the sources that were connected to the primaries, scaled by the transformer ratio. By drawing the voltage sources the other way up, the crossing of connections is avoided.





In this case *the load gets exactly the same voltages, currents and powers as without the open-circuit!*

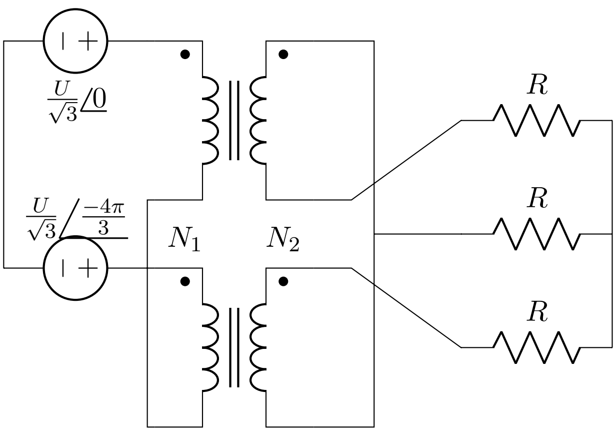
The answer is therefore the same as for part 'b': the source delivers

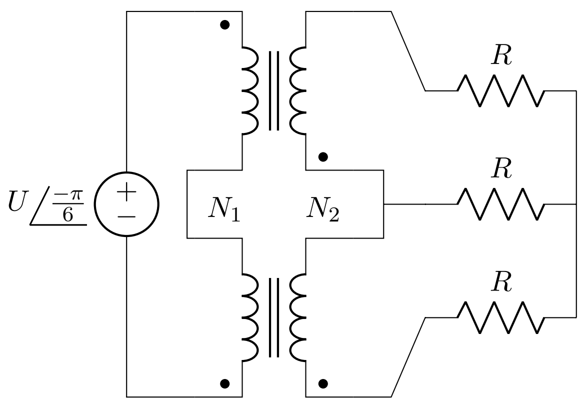
$$\frac{U^2\,N_2^2}{3\,R\,N_1^2}.$$

Reasoning: if we look at voltages between the pairs of line-conductors going to the load, two of these are the voltages across transformer secondaries that still have the same voltage as in the balanced case, because the primaries are still connected directly across phases of the source. The sum of these two voltages appears between the remaining pair of line-conductors. As these voltage were part of a balanced three-phase set, their sum equals the voltage that the now-missing transformer secondary would have had. So in fact, the voltage seen by each phase of the load remains the same. The load power must therefore be the same, and so the power supplied by the source must also, as the circuit has no other component where power can be supplied or absorbed. (The power in each of the two remaining phases of the source must increase so that each produces 50% more than before.)

2. *If there is NOT a neutral from the source to the primaries:*

Again, we remove the middle transformer, and one phase of the source thereby becomes disconnected. This is shown on the left, after removal of open or disconnected components and connections. Without the neutral at the primary side, the two remaining phases of the source are in series, connected across the transformer primaries, which are also in series.





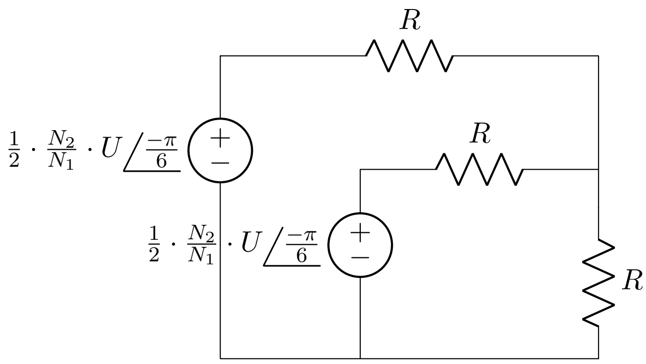
The diagram on the right is after drawing some of the windings the other way up in order to simplify the 'nodal spaghetti', and combining the two series sources into one, using the relation

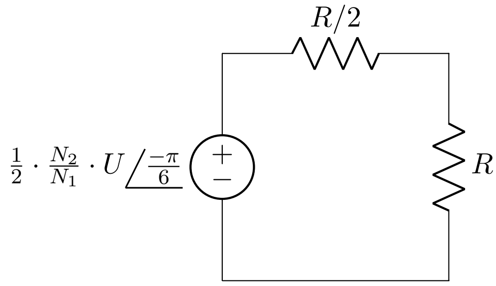
$$\frac{U}{\sqrt{3}}\angle 0 - \frac{U}{\sqrt{3}}\angle\frac{-4\pi}{3} = U\angle\frac{-\pi}{6}.$$

This is a standard relation, showing that the difference between two phase-voltages of a star-connected source gives a line-voltage. The angle is not needed in this solution, but is included here for completeness.

By KCL on the primary side, the current into the dotted-end in the upper transformer must equal the current out of the dot on the lower transformer. The transformers have the same ratio, so in view of the equal-and-opposite primary currents, the transformer equation demands that the currents going into the dots on the secondary side are likewise equal and opposite, but scaled by N_1/N_2. By KCL, the upper and lower resistor carry these same currents, and the middle resistor carries twice this (both currents together).

The transformer secondaries can be represented as in the below left figure, as voltage-sources where half of the primary source voltage is transformed by N_2/N_1 on each secondary. *Warning!* – this replacement of secondaries by voltage sources was rigorous in the previous case with the neutral conductor present, because each primary was fed directly by a voltage source. In the present case, however, the simplified circuit below is only valid because of the symmetry of the circuit on the secondary side. If the horizontal resistors had different values, the primary voltage could divide non-evenly between the two primary windings, so there would not be the simple factor $\frac{1}{2}$ in each secondary voltage.





As the points above the sources in the left diagram have the same potential, by symmetry, we could combine the resistors in parallel to one such source, as shown on the right.

The complex power from the source, equal to the complex power to the resistors, is then

$$\frac{\left(\frac{1}{2}\frac{N_2}{N_1}U\right)^2}{\frac{3}{2}R} = \frac{U^2\,N_2^2}{6\,R\,N_1^2},$$

which is half as much as in the original balanced three-phase circuit or the case where one transformer-winding is broken but a neutral connection is used from the sources to the primaries.

If the rather graphical-thinking solution above is not appealing, you could go back to the original diagram that defines many voltages and currents, and study the case where the neutral is not present and one winding is open so that for example $i_{2\text{b}} = 0$.

Then write the knowledge of this circuit using the familiar rules, not worrying about specific three-phase methods. For example, six transformer-equations relate the voltages and currents of the primary and secondary windings. KVL on the primary side gives $-u_{\text{c}} + u_{\text{a}} = -u_{1\text{c}} + u_{1\text{a}}$. Nodal analysis on the secondary side would define one node as a reference, e.g. the load's star-point. Then algebra, rather than the graphical thinking, can be used to get a solution.