KTH EI1120 Elkretsanalys (CENMI) KS 1 — 2021-02-05 kl. 08–10

Permitted material: Beyond writing-equipment, a single piece of paper up to A4 size can be brought, with free choice of content: handwritten, printed; small, large; text, diagram, image; one or both sides, etc. This paper does not need to be handed in with the exam.

Unless it is stated otherwise, the final answer to a question should be expressed in terms of the known quantities given in the question, and any clear simplifications should be done. Component values such as R for a resistor, U for an independent voltage source, or K for a dependent source, are assumed to be known quantities. Marked currents or voltages such as i_x are assumed to be definitions, not known quantities.

Clearly drawn and labelled diagrams are a good way to help yourself avoid mistakes, and to make clear to others what you are doing. By showing clearly your intermediate steps in a solution, you improve your chance of getting points even if the final result is wrong. You may write in Swedish or English; but we suggest that writing in either is seldom necessary if you make good use of diagrams and equations!

KS1 does not give any direct grade. Its points will be used to replace Section-A in the final exam or re-exam, if this would improve your points there. See therefore the rules for the exam to relate the points to grades: at least 40% is needed in Section-A alone, as well as 50% overall.

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1) [4p]

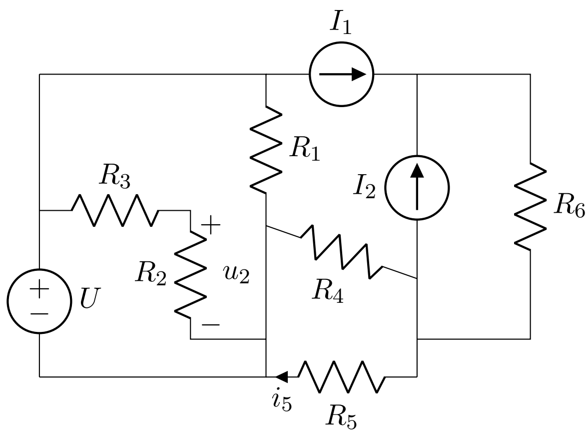
Determine:

a) [1p] the marked voltage u_2

b) [1p] the marked current i_5

c) [1p] the power out of source I_2

d) [1p] the power out of source U



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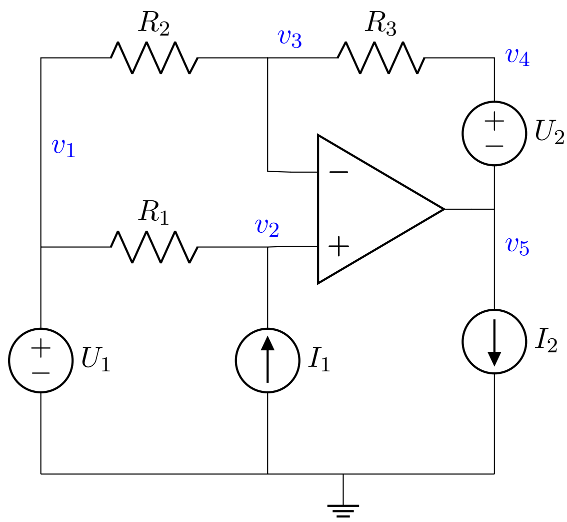
2) [4p]

Write expressions from which the node potentials v_1, v_2, v_3, v_4, v_5 could be determined.

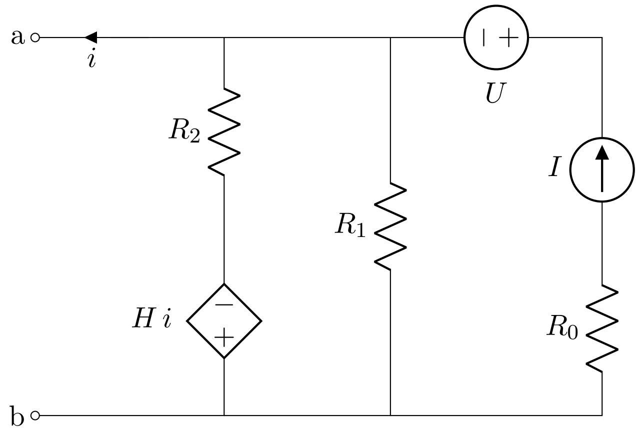
These expressions could be solutions for the potentials, or a set of equations that could be solved to find the potentials. You do not have to simplify or solve any equations. It is simply required that you provide enough information to allow the potentials to be determined in terms of the circuit's component values, using just your expressions (i.e. without needing to look at the circuit diagram).



3) [4p]

a) [2p] What resistance should be connected between terminals ‘a’ and ‘b’ in order to obtain the maximum possible power from this circuit?

b) [2p] What will be the voltage of terminal ‘a’ relative to terminal ‘b’ *when* the resistance chosen in the above (question 3a) is connected?



Översättningar:

Hjälpmedel: Ett A4-ark (båda sidor) med studentens egna anteckningar på valfritt sätt: handskrivet eller datorutskrift; text, diagram, bild; stor eller liten textstorlek, o.s.v. Det måste inte lämnas in med skrivningarna.

Om inte annan information anges i ett tal ska: komponenter antas vara ideala; angivna värden av komponenter (t.ex. R för ett motstånd, U för en spänningskälla, K för en beroende källa) antas vara *kända* storheter; och andra markerade storheter (t.ex. strömmen markerad i ett motstånd eller spänningskälla) antas vara *okända* storheter. Lösningar ska uttryckas i kända storheter och förenklas.

Var tydlig med diagram och definitioner av variabler. Du får skriva på svenska eller engelska, men vi rekommenderar att diagram och ekvationer används i stället i de flesta fall.

KS1 ger inte direkt betyg, utan poäng som kan ersätta poängen i sektion-A i tentan (TEN1, mars) om KS:en gav mer. Se därför reglerna för TEN1 angående gränser.

1. Bestäm följande storheter:

a) [1p] den markerade spänningen u_2

b) [1p] den markerade strömmen i_5

c) [1p] effekten levererad från källan I_2

d) [1p] effekten levererad från källan U.

2. Skriv uttryck från vilka de markerade potentialerna v_1, v_2, v_3, v_4, v_5 skulle kunna bestämmas utan behov av mer information om kretsen. Du kan skriva lösningar för potentialerna, men du kan lika väl skriva ekvationer som skulle kunna lösas. Du får skriva ekvationer utan att behöva lösa eller förenkla dem.

3.

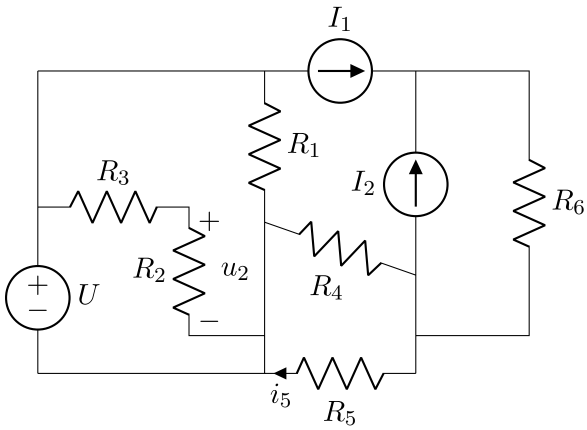
a) [2p] Vilket värde av motstånd ska kopplas mellan polerna ’a’ och ’b’ om den maximala möjliga effekten ska extraheras från kretsen?

b) [2p] När ett motstånd enligt lösningen ovan (deltal 3a) är kopplad mellan polerna, hur mycket är spänningen av pol ’a’ relativ till pol ’b’?

The End. *Don’t waste remaining time . . . check your solutions!*

Solutions (EI1120 KS 1 VT21, 2021-02-05)

Q1.



a) $u_2 = U\frac{R_2}{R_2+R_3}$.

This follows from KVL in the bottom-left loop, and voltage division of U across the two resistors.

b) $i_5 = I_1\frac{R_4}{R_4+R_5}$.

This follows from KCL (around, for example, I_2 and R_6) and current division of I_1 through these two parallel resistors.

c) $P_{_{\text{I2}}} = I_2\,(I_1 + I_2)\,R_6$.

To find the power delivered by this source, we find the voltage across it.
Source I_2 is parallel with R_6, so by KVL they have the same voltage.
The current downwards through R_6 can be found as $(I_1 + I_2)$ by KCL at its top node.
So, by Ohm's law and the above KCL and KVL, the voltage across source I_2 is $(I_1 + I_2)\,R_6$, which is the voltage of its top terminal relative to its bottom terminal. This voltage is in the 'active convention' with regard to current I_2, so their product gives the power delivered from this source, as requested.

d) $P_{_\text{U}} = U\left(\frac{U}{R_1} + \frac{U}{R_2+R_3} + I_1\right)$.

To find the power delivered by this source, we find the current through it.
From KCL at the node above this source, there are three branches in which the source's current can flow: it can go in R_3, R_1 and I_1.
By KVL, the source's voltage U is directly applied across R_1 and across the series pair of R_3 and R_2, so the currents in these are easily determined by Ohm's law. The current I_1 is already determined.
Thus, the current out from the '+' terminal of source U is $\frac{U}{R_1} + \frac{U}{R_2+R_3} + I_1$.
The product of this current and the source's voltage gives the source's output power, as this current is defined out of the terminal where voltage is +.

Q2.

I. Extended nodal analysis ("the simple way")

There are three currents that can't directly be expressed by nodal potentials and resistances. Define the current downwards in U_1 as i_α, likewise in U_2 as i_β, and the current out from the opamp's output as i_o.

Taking KCL at every node except the reference (earth) node:

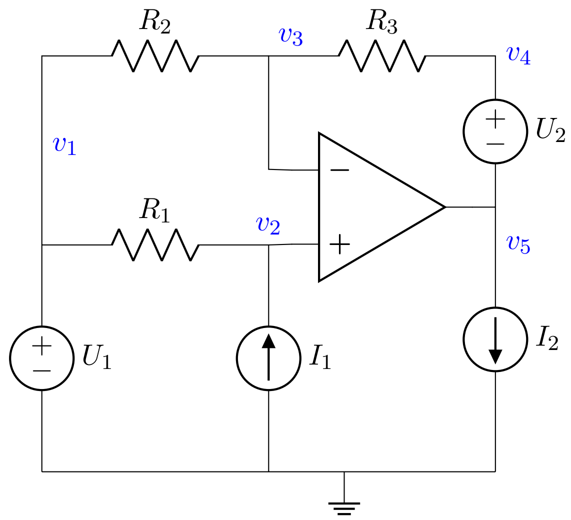
$$\begin{aligned}
\text{KCL}(1)_{(\text{out})}: \quad 0 &= i_\alpha + \frac{v_1 - v_2}{R_1} + \frac{v_1 - v_3}{R_2} && (1)\\
\text{KCL}(2)_{(\text{out})}: \quad 0 &= \frac{v_2 - v_1}{R_1} - I_1 + 0 && (2)\\
\text{KCL}(3)_{(\text{out})}: \quad 0 &= \frac{v_3 - v_1}{R_2} + \frac{v_3 - v_4}{R_3} + 0 && (3)\\
\text{KCL}(4)_{(\text{out})}: \quad 0 &= \frac{v_4 - v_3}{R_3} + i_\beta && (4)\\
\text{KCL}(5)_{(\text{out})}: \quad 0 &= I_2 - i_\beta - i_o && (5)
\end{aligned}$$

The two voltage sources gave us unknown currents i_α and i_β, but also give equations relating node potentials:

$$\begin{aligned}
v_1 - 0 &= U_1 && (6)\\
v_4 - v_5 &= U_2 && (7)
\end{aligned}$$

The opamp gave us the unknown current i_o, but also a further equation relating node potentials (at its inputs):

$$v_2 - v_3 = 0. \qquad (8)$$



The above are 8 equations in 8 unknowns: $v_{1..5}, i_\alpha, i_\beta, i_o$.
Because they were found in a systematic way, we trust they are sufficient (linearly independent).

II. Nodal analysis by simplifications including supernodes

With several voltage sources (including the opamp's output) we much reduce the number of KCLs when using the supernode method.

Node 1 (v_1) becomes part of the reference supernode. Its potential can be written as $v_1 = U_1$.

Node 5 (v_5) also becomes part of the reference supernode, as it is connected to the opamp output. Its potential is not known, so we'll still call it v_5.

Node 4 (v_4) is joined to node 5 by U_2, so it also becomes part of the reference supernode. We can express v_4 as $v_5 + U_2$ in order to minimize the number of unknowns in our equations.

All of the above three nodes are treated as part of a single supernode. Because this supernode includes the reference node, we don't do KCL on it.

There now remain just nodes 2 and 3 at which we should write KCL.

We can note, before writing the KCL equations, that the opamp itself tells us that $v_2 = v_3$. Thus, we can use just one of these unknowns in the KCLs, besides v_5. Let's choose v_3.

$$\text{KCL}(2)_{(\text{out})}: \quad 0 = \frac{v_3 - U_1}{R_1} - I_1 \tag{1}$$

$$\text{KCL}(3)_{(\text{out})}: \quad 0 = \frac{v_3 - U_1}{R_2} + \frac{v_3 - v_5 - U_2}{R_3} \tag{2}$$

The above can give a solution for v_3 and v_5, but it's important to provide clear equations from which all the unknowns can be found: we need also

$$v_1 = U_1 \tag{3}$$

$$v_2 = v_3 \tag{4}$$

$$v_4 = v_5 + U_2 \tag{5}$$

III. Non-systematic solution: simplifications etc

This circuit is simpler than many that are used for this type of 'write the equations' question. It is able to be solved by a step-by-step method. We could write equations like this:

$v_1 = U_1$
'by inspection'.

$v_2 = U_1 + R_1 I_1$
by KCL at v_2, Ohm's law in R_1, and KVL (or 'potential change') when going through U_1 and R_1.

$v_3 = v_2$
because these are inputs of an ideal opamp with negative feedback.

$v_4 = v_3 + \frac{R_3}{R_2}(v_3 - v_1)$
by rearranging KCL at node 3.

$v_5 = v_4 - U_2$
KVL, perhaps better described as 'potential-change'.

Or one write the above equations with substitution of already-calculated potentials into later equations, so that all potentials are expressed only in terms of the given quantities ('known' variables). But doing this is *not* a requirement of the question.

$v_1 = U_1$

$v_2 = U_1 + R_1 I_1$

$v_3 = U_1 + R_1 I_1$

$v_4 = (U_1 + I_1 R_1)\left(1 + \frac{R_3}{R_2}\right) - \frac{R_3}{R_2} U_1$

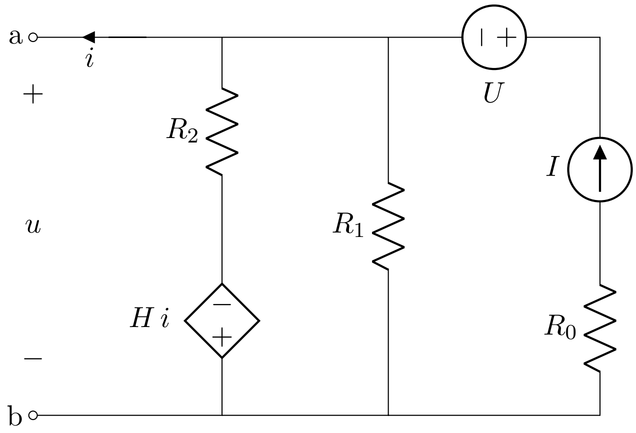
$v_5 = (U_1 + I_1 R_1)\left(1 + \frac{R_3}{R_2}\right) - \frac{R_3}{R_2} U_1 - U_2$

Q3.

The subquestions want us to find what resistance should be connected to this circuit in order to extract maximum power from it (the same resistance as the circuit's Thevenin resistance) and what voltage there is at the terminals in this situation (half of the circuit's Thevenin voltage).

It's therefore clearly going to be useful to find the Thevenin equivalent of this circuit at terminals 'a' and 'b'.

Below are two methods to find the Thevenin equivalent.



I. Relate terminal quantities by equation

One method of finding the Thevenin (or Norton) equivalent is to find an equation that relates the voltage and current at the terminals. These are marked as u and i in the diagram above.

Notice that the branch on the right behaves like a current-source I: the other components U and R_0 have no effect on what happens outside this branch, as the branch must have a fixed current I (by KCL and the nature of the current source) and its total voltage will be determined by what other things are connected to it. So we can start by ignoring U and R_0.

We have a circuit with 4 parallel branches, one of which is the terminals with their unknown current and voltage (unknown because we haven't yet decided what's connected there).

Writing KCL at the node on terminal 'a':

$$i + \frac{u + H\,i}{R_2} + \frac{u}{R_1} - I = 0$$

$$i\left(1 + \frac{H}{R_2}\right) + u\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - I = 0$$

which can be written in the form $u = U_{\mathrm{T}} - R_{\mathrm{T}}\,i$, as

$$u \;=\; \frac{IR_1R_2}{R_1+R_2} - \frac{R_1R_2\left(1+\frac{H}{R_2}\right)}{R_1+R_2}\,i \;=\; \frac{IR_1R_2}{R_1+R_2} - \frac{R_1\,(R_2+H)}{R_1+R_2}\,i$$

revealing that

$$U_{\mathrm{T}} = \frac{IR_1R_2}{R_1+R_2} \qquad \text{and} \qquad R_{\mathrm{T}} = \frac{R_1\,(R_2+H)}{R_1+R_2}.$$

Another way to obtain this is to transform the series branch of $H\,i$ and R_2 to its Norton equivalent, which is a downward-pointing current source $\frac{H}{R_2}\,i$ in parallel with a resistor R_2. Then the circuit is five parallel branches, of which two are resistors, two are current-sources (one dependent, one independent) and one is a marked current i. This leads to the same KCL as the above.

II. Open-circuit voltage and Short-circuit current

Another way to find the Thevenin equivalent is to consider the short- and open-circuit conditions. We know the short-circuit current equals the Norton equivalent's current-source, and the open-circuit voltage equals the Thevenin equivalent's voltage source.

Open-circuited terminals: this condition means that $i = 0$.
This in turn means that the dependent voltage source is fixed to zero, so it behaves as a short-circuit and its branch becomes just R_2.
By KCL at the top node, all the current from source I must pass down through R_1 and R_2 in parallel, as $i = 0$. So the voltage across these resistors, which is also the voltage u, is the product of I and the parallel resistance:

$$u_{(\mathrm{oc})} = \frac{IR_1R_2}{R_1+R_2}$$

Short-circuited terminals: this condition means that $u = 0$.
This means that no current passes in R_1, by KVL and Ohm's law.
A current $\frac{H\,i}{R_2}$ passes down in R_2, by KVL and Ohm's law.
KCL in the top node gives

$$i + \frac{H\,i}{R_2} + 0 - I = 0,$$

from which

$$i_{(\mathrm{sc})} = \frac{I}{1+\frac{H}{R_2}}.$$

Therefore,

$$U_{\mathrm{T}} = u_{(\mathrm{oc})} = \frac{IR_1R_2}{R_1+R_2} \qquad \text{and} \qquad R_{\mathrm{T}} = \frac{u_{(\mathrm{oc})}}{i_{(\mathrm{sc})}} = \frac{\frac{IR_1R_2}{R_1+R_2}}{\frac{I}{1+\frac{H}{R_2}}} = \frac{R_1\left(R_2+H\right)}{R_1+R_2},$$

which is as found by method **I** above.

Having found the Thevenin equivalent, the answers are easily obtained from the maximum power theorem:

a) A resistance of

$$\frac{R_1\left(R_2+H\right)}{R_1+R_2} \qquad \text{or equivalently} \qquad \frac{R_1R_2\left(1+\frac{H}{R_2}\right)}{R_1+R_2}$$

should be connected in order to extract the maximum possible power from the terminals of this circuit.

b) In the above situation, the terminal voltage will be half of its open-circuit value,

$$u = \frac{U_{\mathrm{T}}}{2} = \frac{IR_1R_2}{2\left(R_1+R_2\right)}.$$