KTH, Electric Circuit Analysis, EI1120

Topic 03: Systematic methods Practice Exercises

This course may use your nodal analysis skills in two distinct ways:

1. As a *step within solving* a circuit.
   The exams don't usually *require* a specific method such as nodal analysis or source transformation. Nodal analysis is *often an efficient method* when faced with a circuit in which you don't see any obviously better path to a solution.
   Using simplifications and nodal analysis together is a *very good idea*, in order to reduce the number of equations.

2. As a *method for writing equations* that describe a circuit.
   In this type of case, it's probably best to use what we've called the extended method, where you don't try any simplifications.
   A traditional exam task in this course was "write equations that *could be solved* for v_1, v_2 etc, but you don't need to solve them". Some examples from old exams are linked below. However, note that in Canvas-based exams there would not be exactly this task, although there could be a task of stating which of several options is the correct one.

Both types of skill above should be practised, but particularly the use of nodal analyis to get a solution. Here follows a suggestion of some exercises to do:

- First, the exercises on the following pages.
  These provide some practice to get started with applying nodal analysis within circuit solutions. Thus, it involves solving the equations, not just writing them.

- Other cases, even if not specifically about nodal analysis!
  For many of the circuits that we've already seen in this course, nodal analysis would be a good solution method. A few further exam questions where nodal analysis is useful – and is suggested in the included solution (facit) – are the following:

  2013-06˙EM˙omtenta Q1

  2014-01˙IT˙tenta Q1b

  2015-10˙IT˙omtenta Q1a finding power in R_1

- Writing equations but not having to solve them.
  The following past-exam questions are good examples of extended nodal analysis, with no simplifications needed:

  2013-01˙EM˙ks Q3

  2014-01˙IT˙tenta Q2

  2014-03˙EM˙tenta Q2

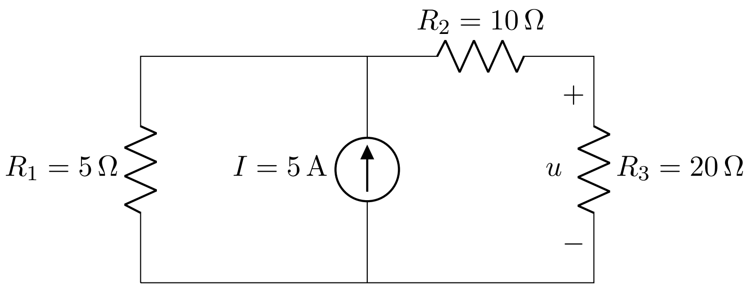
  2014-05˙EM˙omtenta Q2

  2015-03˙EM˙tenta Q2

  Common errors are wrong signs in KCL equations, KCL at the wrong nodes, missing a component from a KCL equation, and confusion about treating a dependent current source as a dependent voltage source or vice versa!

Exercise 1

Use nodal analysis to find the voltage u in the following circuit.
You could check your answer by an alternative method: e.g. current division.

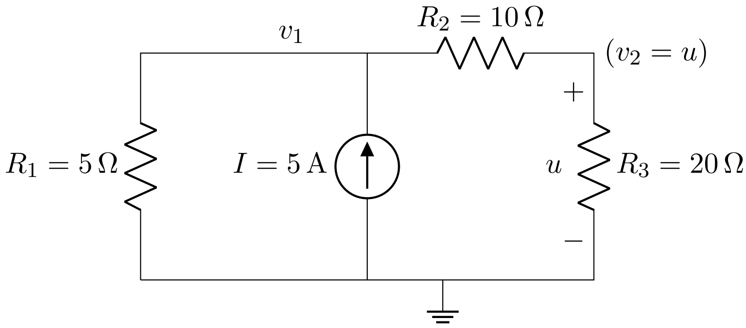


Answer 1

$u = 14.29\,\mathrm{V}$

In order to do nodal analysis, we mark a reference potential (earth). This could be any node. It is good to choose one with many connections. We make the conventional choice here of the bottom node.

Then node potentials must be defined. There are only two other nodes. We can initially try plain nodal analysis without doing simplifications; in that case all the other nodes must have potentials defined: let's mark v_1 and v_2. In this case, the definition of u, and our chosen position of the earth node, result in $v_2 = u$.



The node equations are KCL at the two nodes. No other equations are needed – there are no voltage sources or dependent sources.

$\mathrm{KCL}(1)_{\mathrm{out}} \quad 0 = \dfrac{v_1}{R_1} + \dfrac{v_1 - v_2}{R_2} - I$

$\mathrm{KCL}(2)_{\mathrm{out}} \quad 0 = \dfrac{v_2 - v_1}{R_2} + \dfrac{v_2}{R_3}$

Since we only really want to find u, which is equal to v_2, the obvious choice is to eliminate v_1 from the above pair of equations. This gives

$$v_2 = u = \frac{R_1 R_3 I}{R_1 + R_2 + R_3}.$$

Substitution of the given values gives

$$v_2 = u = \frac{5\,\Omega \cdot 20\,\Omega \cdot 5\,\mathrm{A}}{5\,\Omega + 10\,\Omega + 20\,\Omega} = 14.3\,\mathrm{V}.$$

* * * *

An alternative method is to treat R_2 and R_3 as a single branch (equivalent resistor) and find just the potential v_1, with a single equation. Then voltage division could be used to find u from v_1.

The KCL equation is simply

$$\mathrm{KCL}(1)_{\mathrm{out}} \quad 0 = \frac{v_1}{R_1} + \frac{v_1}{R_2 + R_3} - I,$$

from which

$$v_1 \left(\frac{1}{R_1} + \frac{1}{R_2 + R_3} \right) = I,$$

and so

$$v_1 = \frac{IR_1 \left(R_2 + R_3\right)}{R_1 + R_2 + R_3}.$$

Voltage division gives $u = \dfrac{R_3}{R_2 + R_3} v_1$.

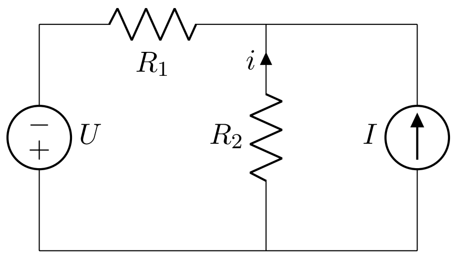
By substituting the equation for v_1 and rearranging,

$$u = \frac{R_1 R_3 I}{R_1 + R_2 + R_3}.$$

That is the same answer as from the first method, but this way felt rather quicker.

Exercise 2

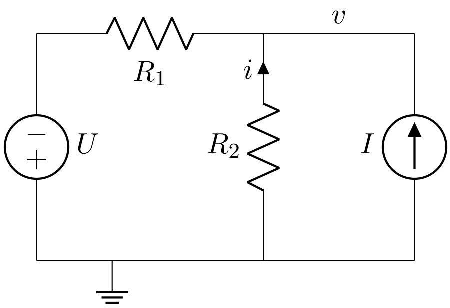
Find the current i, by nodal analysis.



Answer 2

Task: find the current i.

Let's set the bottom node to be the ground node. We only need to find the potential of the node above resistor R_2 in order to define i; we'll call this potential v.



If a complete KCL equation can be found at the node v, without introducing extra unknowns, then v can be determined. This problem is the familiar type of case where there are several branches between a marked node and ground: we just need to find the current in each branch in terms of v, in order to write KCL.

$$\mathrm{KCL}(v)_{\mathrm{out}}: \qquad \frac{v-(-U)}{R_1} + \frac{v}{R_2} - I = 0$$

$$v = \frac{I - \frac{U}{R_1}}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{(IR_1 - U)\,R_2}{R_1 + R_2}$$

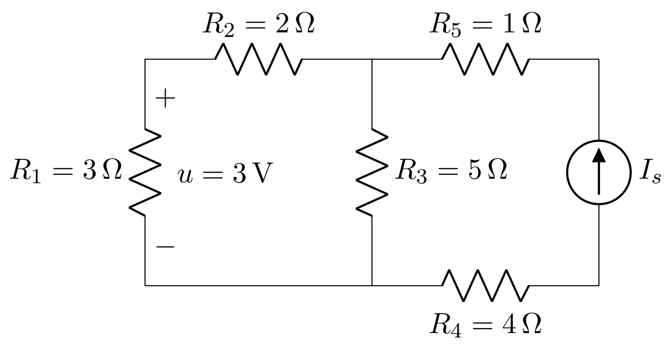
The sought quantity was actually i, not v.

Since $i = -\frac{v}{R_2}$, our final answer is

$$i = \frac{U - IR_1}{R_1 + R_2}.$$

Exercise 3

Find the value of the current source I_s in the following circuit:



This is a backward question ('design question') where a component value needs to be found, in order to give a desired value of some other quantity in the circuit.

It should be quite easy to check your answer by using a few steps of KCL and KVL to confirm that the calculated current makes sense.

Answer 3

$I_s = 2\,\mathrm{A}$

One of the component values (I_s) is unknown, but a marked quantity ($u = 3\,\mathrm{V}$) is known. This does not need to change how we write the nodal equations, although it will affect how we have to rearrange them at the end.

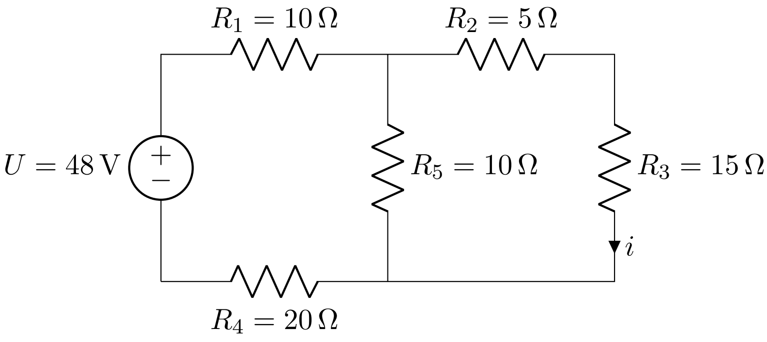
There are several ways that this circuit could be quickly solved by simplifications before and/or after nodal analysis, or even without formally thinking of nodal equations. Try finding some ways. For example, nodal analysis between top and bottom (earth) of R_3, with three simplified branches.

CHECK! Having calculated $I_s = 2\,\mathrm{A}$, let's see if this fits with the desired $u = 3\,\mathrm{V}$. Notice that R_4 and R_5 are irrelevant: the 2 A current comes out of the right branch regardless of their values, and splits (divides) between the middle and left branches. These middle and left branches each are equivalent to a $5\,\Omega$ resistance, so half of the current flows in each of them (current division: it should become a habit to see that two equal branches means halving of the current). By Ohm's law, $u = 3\,\Omega \cdot \frac{2\,\mathrm{A}}{2}$, which is $3\,\mathrm{V}$. That fits with what we wanted.

EXAMPLE (of an alternative non-nodal method). R_4 and R_5 are irrelevant because they are in series with a current source: the complete branch of R_4,I_s,R_5 behaves like a current source I_s towards the rest of the circuit. The series resistors $R_1 + R_2$ have equal resistance ($5\,\Omega$) to the parallel-connected resistor R_3, so current-division and Ohm's law tells us that $\frac{u}{R_1} = \frac{I_s}{2}$, i.e. $I_s = \frac{2 \cdot 3\,\mathrm{V}}{3\,\Omega} = 2\,\mathrm{A}$.

Exercise 4

Find the current i in the following circuit:



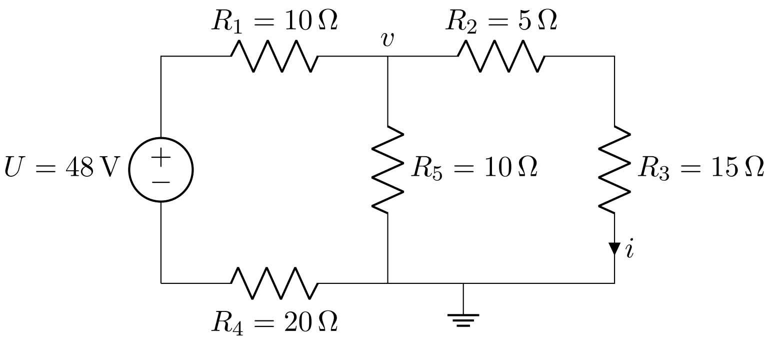
Answer 4

$i = 0.44\,\mathrm{A}$

We'll work on the principle of "use simplifications where possible, unless you have a computer to do the algebra"!

If we mark an earth node and a single other potential v, as shown below, then we have a circuit with just three branches between v and earth. Solving for v then allows i to be found easily by Ohm's law.

Why not another choice of node v — for example, above R_3, to give an even simpler use of Ohm's law to find i? The reason is that there is then not such a simple way to write KCL: the circuit to the left of this point does not have an obvious way of behaving as a single branch (try!).



At node v, KCL gives

$$\mathrm{KCL}(v)_{\mathrm{out}}: \qquad \frac{v-U}{R_1+R_4} + \frac{v}{R_5} + \frac{v}{R_2+R_3} = 0$$

which gives v as

$$v = \frac{U}{(R_1+R_4)\left(\frac{1}{R_1+R_4} + \frac{1}{R_5} + \frac{1}{R_2+R_3}\right)}$$

We have to admit that this is expression is a bit nasty-looking; it can be manipulated further to try to simplify it, but without great improvement. When actual values are available to us, it can be worth including them to reduce the long expressions to single numbers. For example, in the language format of `Matlab` or Octave,

```
U=48, R1=10, R2=5, R3=15, R4=20, R5=10
v = U / ( (R1+R4) * ( 1/(R1+R4) + 1/R5 + 1/(R2+R3) ) )
   8.7273
i = v/(R2+R3)
   0.43636
```

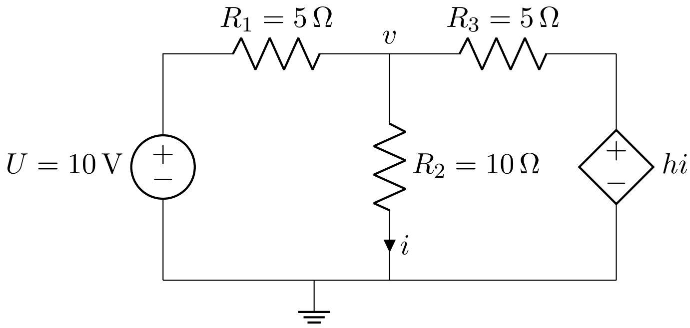
That is the solution for i, using the given numbers. You might have used a calculator instead. Alternatively, if working symbolically, the expression for v would be substituted into $i = \frac{v}{R_2+R_3}$.

Exercise 5

The "component value" of the dependent source (CCVS) is defined here as h.

Write the KCL equation for the node marked v, in terms of the component values.

Now assume that $v = 5\,\mathrm{V}$: what must h be in this case?



Answer 5

$h = 5\,\Omega$

The controlling variable i can immediately be written as $\frac{v}{R_2}$.

By the supernode approach (or equivalently, by treating the circuit as three branches), one KCL equation can be written,

$$\mathrm{KCL}(v)_{\mathrm{out}}: \qquad \frac{v-U}{R_1} + \frac{v}{R_2} + \frac{v - h\frac{v}{R_2}}{R_3} = 0.$$

Rearranging,

$$vR_1\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} - \frac{h}{R_2R_3}\right) = U.$$

We know all values except h. Rearranging in terms of h,

$$h = R_2R_3\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} - \frac{U}{vR_1}\right)$$

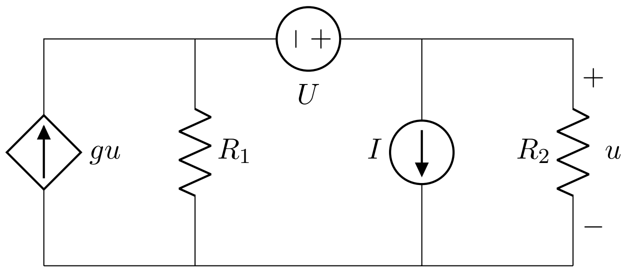
Substituting the given numbers,

```
U=10, R1=5, R2=10, R3=5, v=5
h = R2*R3 * ( 1/R1 + 1/R2 + 1/R3 - U/(v*R1) )
```

Exercise 6

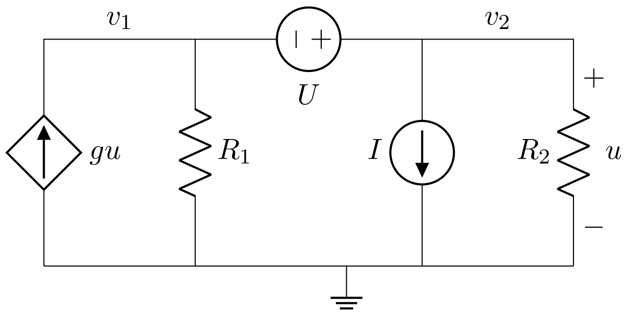
Find the marked voltage u.

[The values of all components are assumed, as usual, to be known. Notice that the current source at the left (a dependent source: in fact a VCCS) has a component value of g, meaning that its current is gu where u is the controlling variable.]



Answer 6

Introduce names for the node potentials, and an earth node.



Let's use the supernode approach. The top two nodes form one supernode. Their potentials are related by $v_1 = v_2 - U$. Let's choose v_2 as the unknown: then we will write $v_2 - U$ instead of v_1. The marked voltage u that is the controlling variable for the VCCS can be defined easily in terms of the other potentials: we see that $u = v_2$.

We only have to write one KCL, for this supernode.

$$\mathrm{KCL(1,2)_{out}} \quad 0 = -gv_2 + \frac{v_2 - U}{R_1} + I + \frac{v_2}{R_2}.$$

This can be solved for v_2, and thus for the requested u,

$$u = v_2 = \frac{U - IR_1}{1 + \frac{R_1}{R_2} - gR_1}.$$

The exercises after this page are not strongly recommended.
That is, unless you want extra challenge, it's probably more efficient at this point to look at some past exams (see front page).

Ex7 is more about power.

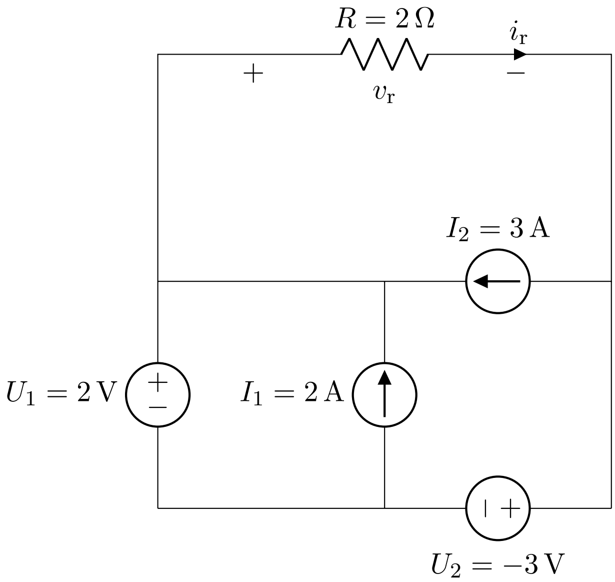
Ex8 and Ex9 are not well checked for their answers.

Ex10 is a little tedious for the algebra if you do it by hand.

See the first page about other sources of examples.

Exercise 7

Find the current i_r, the voltage u_r, and all the powers in the following circuit:



This is a case that can also be handled quite easily by direct use of KCL and KVL, without following any particular nodal-analysis rules. But try starting with the supernode method, and see how simple it becomes. Check your results.

Answer 7

If we try the supernode method of nodal analysis, we find that all three nodes form one supernode, because all nodes are linked through voltage-sources. If we define one node as earth, then all potentials are known, because of the voltage sources. So this is a trivial case, but it's helpful for forcing us to mark the nodes and to make in the values of the voltages.

This voltage and current are already marked on the diagram,

$i_{\mathrm{r}} = 2.5\,\mathrm{A}$

$v_{\mathrm{r}} = 5\,\mathrm{V}$

For the powers, let's calculate for each component the power *into* it from the circuit.

$P_{\mathrm{r}} = 12.5\,\mathrm{W}$

$P_{\mathrm{U1}} = 5\,\mathrm{W}$

$P_{\mathrm{I1}} = -4\,\mathrm{W}$

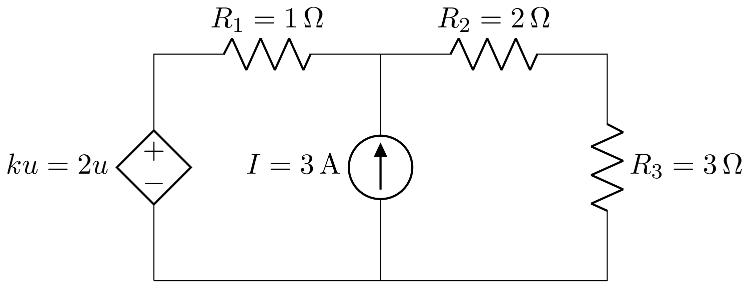
$P_{\mathrm{I2}} = -15\,\mathrm{W}$

$P_{\mathrm{U2}} = 1.5\,\mathrm{W}$

Quick check: fortunately, these values sum to zero. They should, as they are the entire set of components that deliver power to or consume power from the circuit; what goes in should also go out.

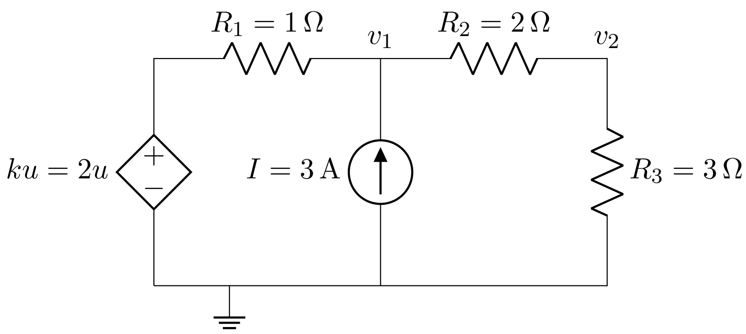
Exercise 8

Find the power delivered by source I.



Answer 8

The power from source I can be found as $v_1 I$ if we mark v_1 and an earth node in the following way:



One way:

KCL at v_1: $\dfrac{v_1 - kv_2}{R_1} - I + \dfrac{v_1 - v_2}{R_2} = 0.$

KCL at v_2: $\dfrac{v_2 - v_1}{R_2} + \dfrac{v_2}{R_3} = 0.$

then substitute to solve v_1.

Another way:

Make the right-hand branch into an equivalent:

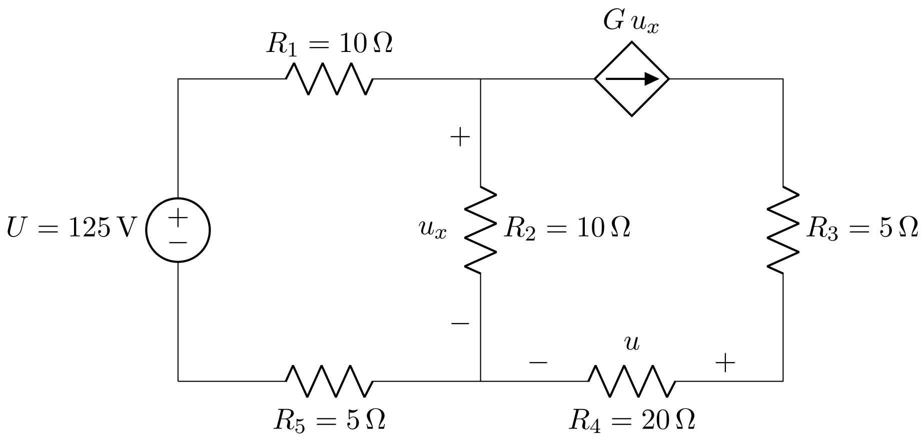
KCL at v_1: $\dfrac{v_1 - kv_2}{R_1} - I + \dfrac{v_1}{R_2 + R_3} = 0.$

Then solve for v_1.

Exercise 9

Find what value G the dependent source in this circuit must have in order to make the marked voltage u be $12.5\,\mathrm{V}$.

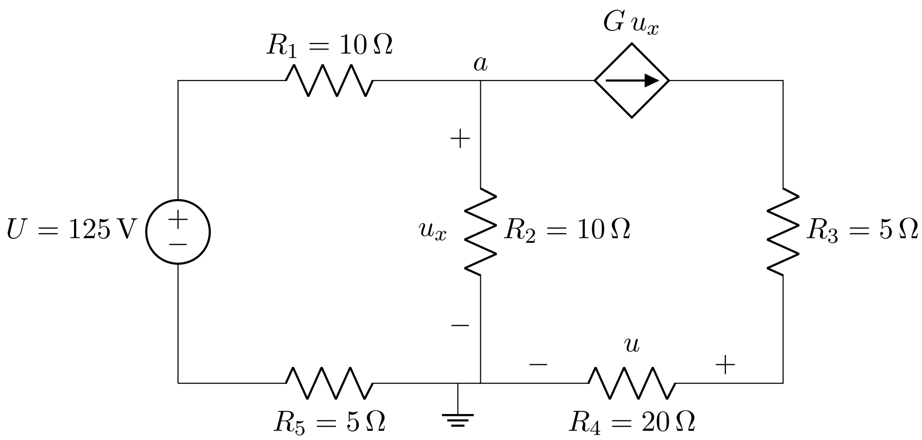
(It's a "backward" question: find a component from a known marked quantity. The same principle can be used: if you find an expression for u as a function of the component values, then you can rearrange it to find one unknown component from a given marked value.)



Answer 9

$G = 0.014\,\mathrm{S}$.

One way of seeing this circuit is as three branches between the top and bottom nodes. We can mark those nodes as potentials zero and a. (In this circuit, this value a is equal to the marked u_x, so we won't really need to mark a or even decide on an earth node; but we do it in order to follow the usual procedure instead of making a special case that might feel confusing.)



The current in R_4 is the same as from the dependent source Gu_x. Therefore, if we can find u_x, we can find u by Ohm's law: $u = R_4 G u_x$. The value u isn't needed in the nodal analysis (it isn't a controlling variable, and we only need KCL at the node 'a'). So we'll start by just looking for u_x.

Bearing in mind $a = u_x$, we can write KCL at a, simplifying the three outgoing branches where possible, e.g. by realising that the current in R_3 and R_4 is independent of these resistors' values.

$$\frac{u_x - U}{R_1 + R_5} + \frac{u_x}{R_2} + G u_x = 0.$$

Substituting $u_x = \frac{u}{R_4 G}$,

$$\frac{u}{R_4 G (R_1 + R_5)} + \frac{u}{R_2 R_4 G} + \frac{u}{R_4} - \frac{U}{R_1 + R_5} = 0,$$

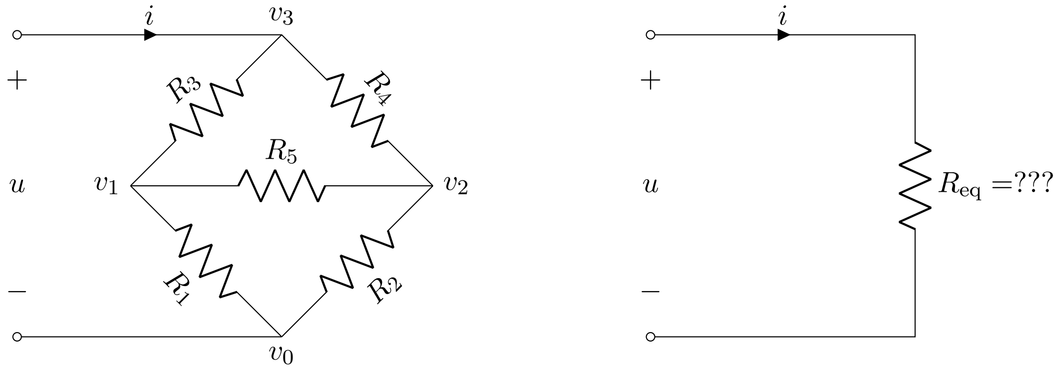
then rearranging for G,

$$G = \frac{\frac{1}{R_4(R_1+R_5)} + \frac{1}{R_2 R_4}}{\frac{U}{(R_1+R_5)u} - \frac{1}{R_4}} = \frac{\frac{1}{R_1+R_5} + \frac{1}{R_2}}{\frac{U R_4}{(R_1+R_5)u} - 1} = \frac{\frac{1}{10\,\Omega+5\,\Omega} + \frac{1}{10\,\Omega}}{\frac{125\,\mathrm{V}\cdot 20\,\Omega}{(10\,\Omega+5\,\Omega)\cdot 12.5\,\mathrm{V}} - 1} = \frac{1}{74\,\Omega} \simeq 0.014\,\mathrm{S}.$$

(The unit S is siemens: $1\,\mathrm{S} = 1\,\Omega^{-1}$. That is expected for the component-value of a VCCS, since it it the ratio of the source's output current to its controlling voltage.)

Exercise 10

Find the value of R_{eq} in order for it to be equivalent to the set of 5 resistors shown in the left. If it helps, assume some specific values of $R_1 = R_4 = 2\,\Omega$, $R_2 = R_3 = 1\,\Omega$ and $R_5 = 3\,\Omega$.

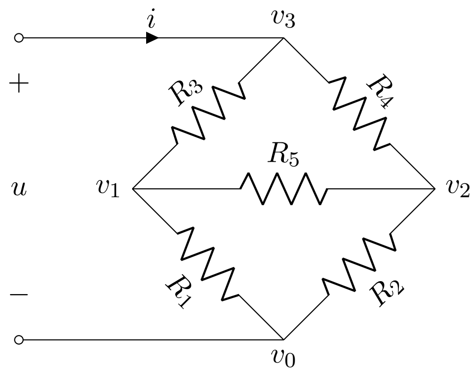


Would there have been an easier approach if $R_1 = R_2 = R_3 = R_4$?

Answer 10

This is a problem where we want to find the relation between u and i.

A good approach is just to define one of these unknowns, then find the other as a function of it. (One way to think of this, in a more physical way than just the equations, is "what if we connect a voltage source across the terminals and measure the current" or "what if we connected a current source and measure the voltage".)



Let's take the first of those methods. We'll do nodal analysis. Define the bottom node as earth: hence $v_0 = 0$. If we set a voltage u at the input, then $v_3 = u$. There are just two unknown potentials, v_1 and v_2. At these nodes, the KCL equations are:

$$\frac{v_1}{R_1} + \frac{v_1 - u}{R_3} + \frac{v_1 - v_2}{R_5} = 0$$

and

$$\frac{v_2}{R_2} + \frac{v_2 - u}{R_4} + \frac{v_2 - v_1}{R_5} = 0$$

We can solve the equations for v_1 and v_2, then calculate the current,

$$i = \frac{u - v_1}{R_3} + \frac{u - v_2}{R_4}.$$

But the solution is very messy with these symbolic variables. Cheating with a computer-algebra program is a nice idea. We wouldn't put questions like this in the exam: we don't want to waste lots of time on manipulation of simultaneous equations. The following is done in the Matlab symbolic toolbox.

```
s= solve( 'v1/r1+(v1-u)/r3+(v1-v2)/r5=0', ...
          'v2/r2+(v2-u)/r4+(v2-v1)/r5=0', ...
          'R=u/((u-v1)/r3+(u-v2)/r4)', ...
       'v1,v2,R')
simplify(s.R)
```

The unpleasant output from this is:

```
 (  r1*r2*r3 + r1*r2*r4 + r1*r2*r5 + r1*r3*r4 + r2*r3*r4
  + r1*r4*r5 + r2*r3*r5 + r3*r4*r5  ) /
 ( r1*r3 + r1*r4 + r2*r3 + r1*r5 + r2*r4 + r2*r5 + r3*r5 + r4*r5 )
```

It's certainly helpful to assume some specific values, such as $R_1 = R_4 = 2\,\Omega$, $R_2 = R_3 = 1\,\Omega$ and $R_5 = 3\,\Omega$. That would make it much easier to solve the simultaneous equations above (try!).

But now that we've already done it the general, symbolic way, and got a long solution, we can substitute the values,

```
r1=2, r4=2, r2=1, r3=1, r5=3
subs(s.R)
```

This gives a solution of $R_{\text{eq}} = 1.44\,\Omega$.

* * * *

Would there have been an easier approach if $R_1 = R_2 = R_3 = R_4$?
Yes! In fact, in any case where $\frac{R_4}{R_2} = \frac{R_3}{R_1}$, the solution is easy. In those cases, the two sides form equal voltage-dividers. The horizontal resistor is therefore connected between points that have identical potential, so no current flows in it: it can be ignored when analysing the circuit's behaviour when seen from the terminals[1] In that case, the circuit is easy: it is two series branches, and these branches are in parallel with each other. Hence, $R_{\text{eq}} = (R_1 + R_3)||(R_2 + R_4)$, where the $||$ symbol means "parallel connection", i.e.,

$$R_{\text{eq}} = \frac{(R_1 + R_3)(R_2 + R_4)}{R_1 + R_2 + R_3 + R_4} \quad \text{(with no } R_5\text{)}.$$

We could also consider some special cases where one or more resistors become very much higher or lower resistance than the others, so that the equivalent can be approximated by a simpler method.

* * * *

Several other methods can be considered, beyond nodal analysis (or mesh analysis, which we don't study). Those that are included in this course are mainly in the topic on Circuit Theorems. The star-mesh transformation (specifically, its simplest case of a wye-delta transformation) would allow one set of three resistors to be replaced with an equivalent but differently connected set, which could then be simplified in series or parallel with the other resistors. The Helmholtz equivalents (e.g. Thevenin or Norton) would allow the two "voltage divider" sides to be modelled as a sources: e.g. remove R_5, then see points v_1-v_0 as a Thevenin equivalent, and similarly $v_2 - v_0$, then reconnect R_5 between these equivalents to find the potentials in the real circuit.

[1] If the horizontal resistor R_5 were *not* connected, then under the condition of equal-ratio voltage dividers at both sides, the voltage between nodes 1 and 2 is $v_2 - v_1 = 0$. If we now connect R_5 between these nodes, we would not expect a current to start flowing in it, since there was no voltage there to push any current. With no current flowing in that branch, it has no effect on the rest of the circuit, so there is no reason that its presence would affect the potentials of the points it connects to; we can assume that the solution without R_5 is the same as the solution with R_5.