KTH, Electric Circuit Analysis, EI1120

Topic 04: Circuit Theorems Practice Exercises

The following examples together with the homework are hoped to be sufficient for understanding this Topic. We expect they're more than enough to fill the available study time for most students.

If you want more practice at superposition, you can *go back* to circuits solved in earlier Topics, where there's more than one independent source, and try superposition for their solution. Then you will get a feeling for whether and when superposition is useful as a solution method.

The principle of Thevenin/Norton equivalents will be used later in the course also, so you will further chances to practice. (But this should also be seen as a hint that it's important to get familiar with finding and using these equivalents right now, so that you can concentrate on the new material in the later Topics.)

The following are some old exam questions about equivalents and/or maximum power. Superposition isn't usually a demand in exam questions: it's a method that might be a useful choice for getting a solution.

See the OldExams page for more, but notice that some questions with equivalents and maximum power include opamps or ac solutions, which we haven't covered yet.

2016-01˙E˙omtenta1 Q3 equivalent circuits, with only independent sources

2016-02˙EM˙ks1 Q3 maximum power

2016-03˙EM˙tenta Q3 Norton equivalent and maximum power

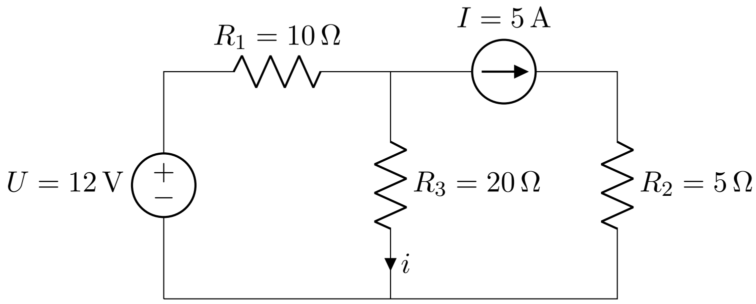
2015-10˙E˙tenta1x Q3 Norton-equivalent in circuit with dependent sources

2015-10˙E˙tenta1 Q3 Norton-equivalent in circuit with dependent sources

Exercise 1

Find the current i using superposition.

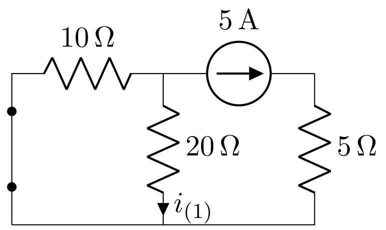
This should be a fairly familiar circuit, but try now to do it without referring again to the lecture notes!



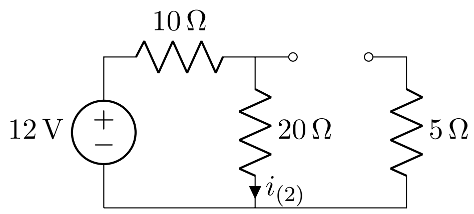
Answer 1

$i = i_{(1)} + i_{(2)} = \dfrac{-5\,\mathrm{A}}{3} + \dfrac{12\,\mathrm{V}}{30\,\Omega} = -1.27\,\mathrm{A}$

Superposition case (1): Zeroing the voltage source.
The current $i_{(1)}$ due to the current source alone can be found by current division. It is $\frac{5\,\mathrm{A}}{3}$, which rounds to $-1.67\,\mathrm{A}$. The factor $\frac{1}{3}$ comes from $\frac{10\,\Omega}{10\,\Omega+20\,\Omega}$. Some of these common current-division relations can be seen quickly without the calculation, when you have used them several times: between parallel resistances R and $2R$, the current will divide with $\frac{2}{3}$ of it flowing in R, and $\frac{1}{3}$ in $2R$.



Superposition case (2): Zeroing the current source.
The current $i_{(2)}$ due to the voltage source alone can be found by an equivalent resistance and Ohm's law. This is $\frac{12\,\mathrm{V}}{10\,\Omega+20\,\Omega}$.

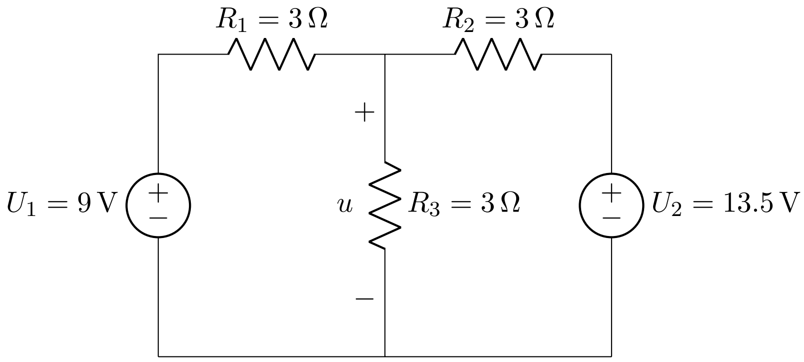


Then, add the two superposition cases together.

Notice that the $5\,\Omega$ resistor did not affect either of the superposition cases (its value does not appear in the equations). Can you see, from the original circuit, that this component is irrelevant to i?

Exercise 2

Find the voltage u using superposition.



You might care to check by comparing with nodal analysis on the complete circuit.

Answer 2

$u = 7.5\,\mathrm{V}$.

With source U_2 set to zero, we have R_2 and R_3 in parallel, with total resistance $\frac{3\,\Omega}{2}$.

This pair is in series with R_1 and U_1.

The current through R_1 is then $\frac{U_1}{R_1+R_2\|R_3}$ where the symbol $\|$ means the parallel combination.

Numerically this is $\frac{9\,\mathrm{V}}{4.5\,\Omega} = 2\,\mathrm{A}$. By current division, half of this current flows down R_3, i.e. $1\,\mathrm{A}$.

In the other superposition state, with U_1 set to zero, we get a similar result except that the higher voltage of U_2 gives a current of $1.5\,\mathrm{A}$ down R_3.

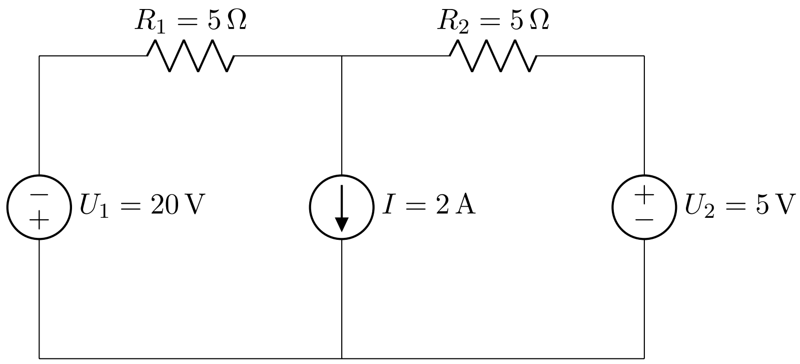
The total current down R_3 is therefore $1\,\mathrm{A} + 1.5\,\mathrm{A} = 2.5\,\mathrm{A}$.

Multiplied by R_3, this current determines the sought voltage:

$$u = 2.5\,\mathrm{A} \cdot 3\,\Omega = 7.5\,\mathrm{V}.$$

Exercise 3

Use superposition to find the power delivered by source I.



Answer 3

Be careful! Power can't be treated by superposition (except when we consider average power from time-varying sources that have 'orthogonal' functions such as sinusoids of different frequencies ... 'power superposition' is in the AC part of the course).

The power depends on squares or other products of voltage and current, like u^2 or i^2 or ui.

We can't just calculate the power with each source active by itself then add these ... bear in mind that $i_1^2 + i_2^2$ is *not* necessarily the same as $(i_1 + i_2)^2$!

Instead, we can use superposition to find a voltage or current, then use this to find the sought power.

If we find the voltage across the current source, this will give us the power that the source delivers. Define u as the voltage of the top node compared to the bottom node. Then the power *delivered* by the current source is $-uI$.

There are three independent sources here.

We could treat each separately, keeping the other two zeroed in each case. Or we could treat some pair of them as active together: here a tempting pair is the two voltage sources since the zeroed current-source results in a single-loop circuit where the voltage u is quite easily found by voltage division and KVL.

But let's treat one source at a time — three superposition states to solve:

U_1 active (only).
Set U_2 as short-circuit, I as open circuit. Redraw the circuit for clarity.
By voltage division, $u_{(1)} = \frac{U_1 R_2}{R_1+R_2} = -10\,\text{V}$.

U_2 active.
Similarly, $u_{(2)} = 2.5\,\text{V}$.

I active.
Here the current source supplies just two parallel resistors: the resulting voltage u across the source is $u_{(3)} = -5\,\text{V}$.

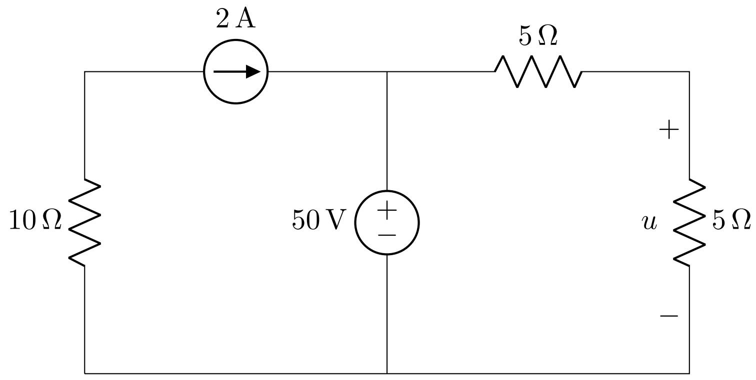
Summing the results, $u = u_{(1)} + u_{(2)} + u_{(3)} = -12.5\,\text{V}$.

The power P delivered by source I is

$$P \;=\; -uI \;=\; -(-12.5\,\text{V}) \cdot 2\,\text{A} \;=\; 25\,\text{W}$$

Exercise 4

Find the voltage u using superposition in the following circuit:

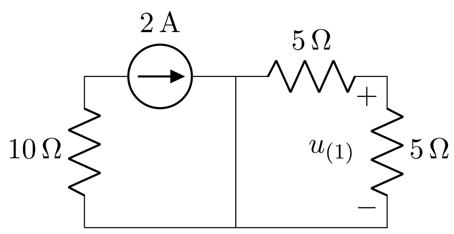


(Can you see the answer quickly, without even needing superposition or nodal analysis or source-transformation? Even if you can, try doing it by superposition, to see how this method handles the problem.)

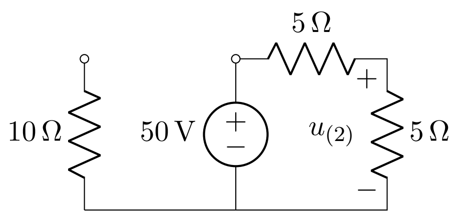
Answer 4

$u = u_{(1)} + u_{(2)} = 0 + \frac{1}{2} \cdot 50\,\mathrm{V} = 25\,\mathrm{V}$

Superposition case (1). Just the current source (zeroed voltage source).
Short-circuit and voltage-division show that $u_{(1)} = 0$.



Superposition case (2). Just the voltage source (zeroed current source).
Use voltage division to find $u_{(2)} = \frac{1}{2} \cdot 50\,\mathrm{V}$.

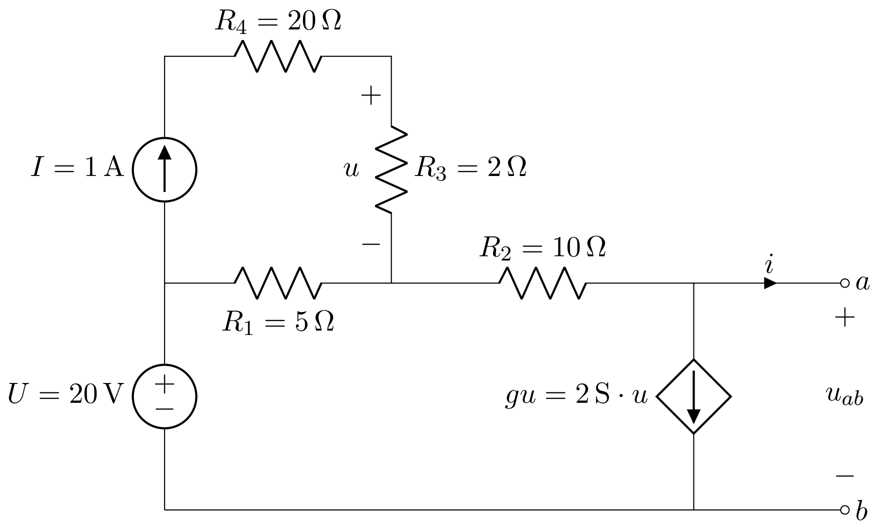


We could avoid bothering with superposition at all by noticing that the current source and $10\,\Omega$ resistor are both irrelevant to u, since the voltage source is in parallel with that branch and with the $5\,\Omega$ resistors. KVL already tells us the voltage across the series pair of resistors, which is not affected by source I. Thus, voltage division could have been used from the start, without doing superposition.

(But the question still gives practice at doing superposition, and at not committing the classic error of choosing the wrong short- or open-circuit for the two different types of source when zeroed!)

Exercise 5

Use superposition to find the open-circuit voltage u_{ab} here:

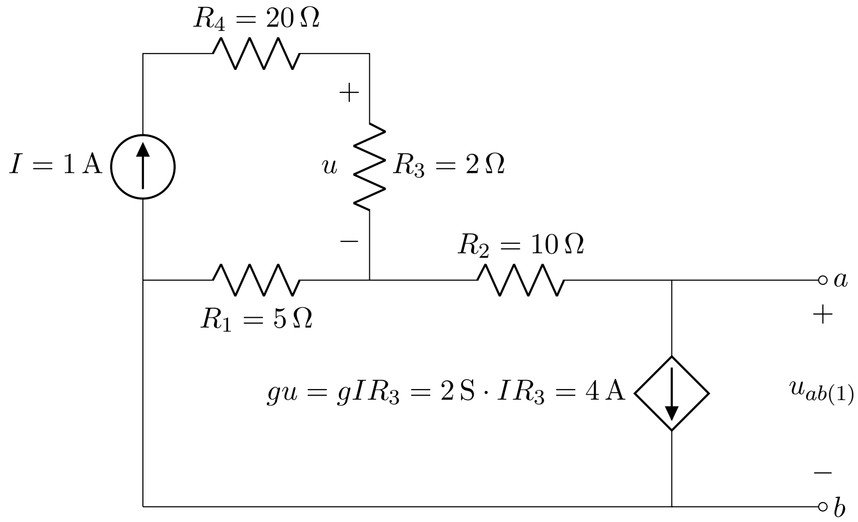


[The dependent current source is described here as $gu = 2\,\text{S} \cdot u$. This simply indicates that the source's value can be called g in our equations, and that the actual value is $2\,\text{S}$ (i.e. two siemens, $2\,\Omega^{-1}$). A voltage-controlled current source naturally has this dimension $\frac{1}{\Omega}$ for its value, as it "converts a voltage to a current".]

Answer 5

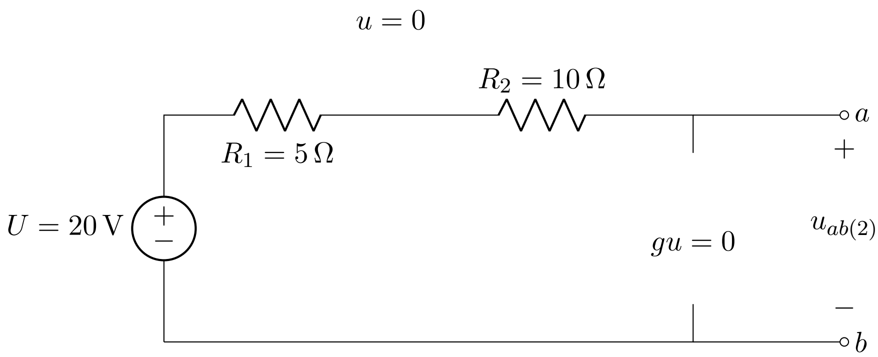
$u_{ab} = -35\,\text{V}$.

Zeroing (short-circuiting) the voltage source:



This is actually not as hard as it might look, *if* seen in the right way! The controlling variable u is fixed by the current source and R_3, so it behaves like an independent source of gIR_3. In the open-circuit state, there is no current from a-b, so KCL in the middle node tells us that the current from left to right in R_1 must be $-1\,\text{A}+4\,\text{A} = 3\,\text{A}$. Taking KVL round a loop clockwise from point b gives $-R_1 3\,\text{A} - R_2 4\,\text{A} - u_{ab(1)} = 0$, which tells us that $u_{ab(1)} = -5\,\Omega \cdot 3\,\text{A} - 10\,\Omega \cdot 4\,\text{A} = -55\,\text{V}$.

Zeroing (open-circuiting) the current source:



By forcing the current in the top loop to zero, the whole top loop can be ignored: this also means that $u = 0$, which means the dependent current source has been forced to zero too, so we can also treat this as an open-circuit. This circuit cannot have any current in with a-b open-circuited, so KVL gives $u_{ab(2)} = U = 20\,\text{V}$.

By superposition, $u_{ab} = u_{ab(1)} + u_{ab(2)} = -55\,\text{V} + 20\,\text{V} = -35\,\text{V}$.

Another way of handling the whole problem quite efficiently is a combination of nodal analysis and simplification. For example, let's write KCL for just the central node (below R_3) in the complete circuit with both independent sources active and the output a-b open-circuited. Let's define the bottom node (e.g. point 'b') as earth, and call the potential of the central node v. Then,

$$\text{KCL}(v)_{\text{out}} \qquad 0 = \frac{v-U}{R_1} + gu - I.$$

This has the trouble that u is not a known variable. But noticing that u is across a resistor that is in series with a current source, we can write that $u = IR_3$, and this can be substituted. Solving for v, and rearranging,

$$v = U + R_1\,(1-2R_3)\,I = 5\,\text{V}.$$

The relation of v to u_{ab} in the open-circuit case is that $u_{ab} = v - R_2 g u$.

Therefore,

$$u_{ab} \;=\; v - R_2\,g\,R_3\,I \;=\; 5\,\text{V} - 10\,\Omega\cdot 2\,\text{S}\cdot 2\,\Omega\cdot 1\,\text{A} \;=\; -35\,\text{V}.$$

More formal application of the supernode or extended method of nodal analysis could be used. This would require less thought about the circuit, but would result in more equations than we really needed for solving our single sought quantity of u_{ab}.

Exercise 6

In the previous question (for which the open-circuit voltage u_{ab} was found) find also the Thevenin and Norton equivalents of the circuit, between terminals a-b.

Answer 6

We have the open-circuit voltage u_{ab}, from the previous question's solution.

If we find the short-circuit current, the Thevenin or Norton equivalent can be determined.

Consider a short-circuit between a-b.
The current through R_2 would now be $\frac{v}{R_2}$ instead of gu, so

$$\text{KCL}(v)_{\text{out}} \qquad 0 = \frac{v-U}{R_1} + \frac{v}{R_2} - I.$$

We can solve this to give $v = \frac{R_2(U+IR_1)}{R_1+R_2}$. Dividing v/R_2 gives the current towards a in R_2. KCL at the right of R_2 gives the current coming out of terminal a into the short-circuit: $i_{\text{sc}} = \frac{v}{R_2} - gu$.

We again have to substitute $u = IR_3$ when finding the dependent source's current.

The result is that

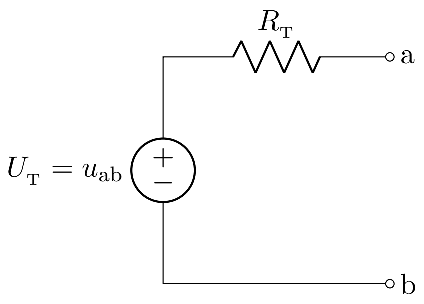
$$i_{\text{sc}} = \frac{U+IR_1}{R_1+R_2} - gR_3I = -2.33\,\text{A}$$

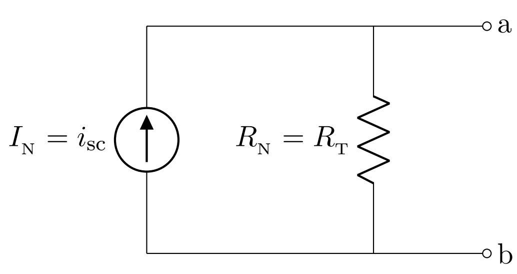
From this, the source-resistance seen at terminals a-b can be found. We'll call it R_{T}, but realising that it could equally well have been R_{N} since Thevenin and Norton resistances are always the same.

$$R_{\text{T}} = \frac{u_{ab,\text{oc}}}{i_{ab,\text{sc}}} = \frac{-35\,\text{V}}{-2.33\,\text{A}} = 15\,\Omega.$$

In this particular circuit, the above result for R_{T} can be seen by 'setting the sources to zero' and looking at the resistance into a-b: just the $10\,\Omega$ and $5\,\Omega$ resistors in series are seen. But be warned – this is a *special case* where the dependent current source is known to be fixed at zero (open circuit) when the independent current source is set to zero. If instead the controlling variable u had been across, for example, R_1, then the dependent source would become relevant to the Thevenin source resistance. So – *in general* – do not use the method of finding source resistance directly, if there are dependent sources in the circuit!

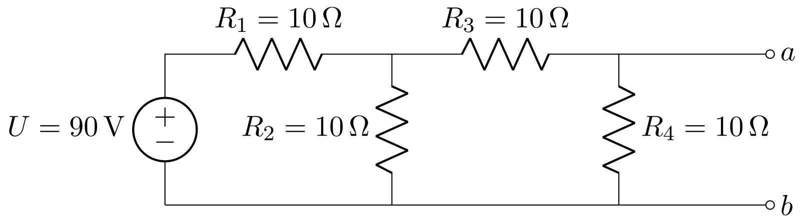
The Thevenin (left) and Norton (right) equivalents are shown below.





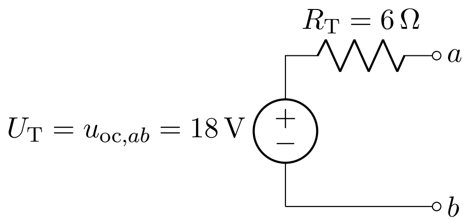
Exercise 7

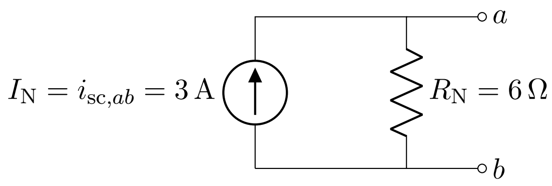
Find the Thevenin and Norton equivalents between a and b in the following circuit:



Answer 7

Thevenin and Norton equivalents:





The marked 'a' and 'b', and the direction of the source, are important details of the equivalent. Just writing the numbers is not enough.

How to find the values:

Open-circuit voltage $u_{oc,ab}$ between 'a' and 'b':
From the perspective of the voltage source, R_2, R_3 and R_4 can be made into a single equivalent resistor of $R_x = \frac{(R_3+R_4)R_2}{R_2+R_3+R_4}$ that replaces R_2 and everything to the right of it.
Then voltage division with R_1 gives a voltage of $\frac{UR_x}{R_1+R_x}$ on that equivalent.
The sought voltage can be found from this by further voltage division between R_3 and R_4: $u_{oc,ab} = \frac{UR_x}{R_1+R_x} \cdot \frac{R_4}{R_3+R_4}$. This becomes $18\,V$ with the given values.

Short-circuit current $i_{sc,ab}$ from 'a' to 'b':
Now R_4 can be ignored as it is short-circuited.
The voltage source sees R_2 and R_3 in parallel, with R_1 in series with them.
The source therefore supplies a current of $\frac{U}{R_1+R_2R_3/(R_2+R_3)}$.
By current division in R_2 and R_3, the short-circuit current is $i_{sc,ab} = \frac{U}{R_1+R_2R_3/(R_2+R_3)} \cdot \frac{R_2}{R_2+R_3}$.
This is $3\,A$.

The equivalent-source resistance could be found from $R_T = R_N = \frac{u_{oc,ab}}{i_{sc,ab}} = \frac{18\,V}{3\,A} = 6\,\Omega$.
That is sensible when we already have been forced to find the short-circuit current and open-circuit voltage.
If we only knew one of those, we could find the resistance more directly by the method suitable for equivalents of a circuit that doesn't have dependent sources: set the sources (U in this case) to zero, and look at the equivalent resistance of the circuit between its terminals, i.e. R_{ab}.
Try this, and confirm that it results in $6\,\Omega$!

Try, instead, nodal analysis to find the short-circuit current and open-circuit voltage!

Exercise 8

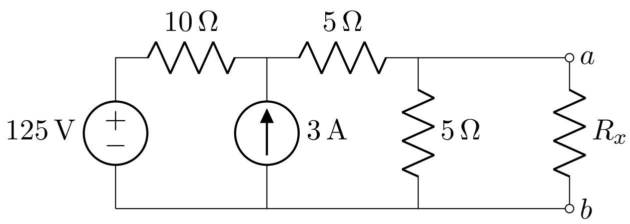
i) Find the value of R_x that causes the maximum possible power transfer from the circuit on the left of terminals 'a' and 'b', into R_x.

ii) What is the maximum power that can be obtained by R_x?

iii) Find the power consumed by resistor R_x for the cases $R_x = 1\,\Omega$ and $R_x = 10\,\Omega$.

iv) If the resistance R_x is replaced by a current source I_x with its arrow pointing upwards, is there a particular value of source current that gives maximum power? If yes, what is that value, and what is the maximum power?

v) Consider part 'iv' again, but instead of a current-source I_x, put a voltage source U_x, with its +-terminal upwards.



Hint: First find the Thevenin or Norton equivalent between nodes a and b.

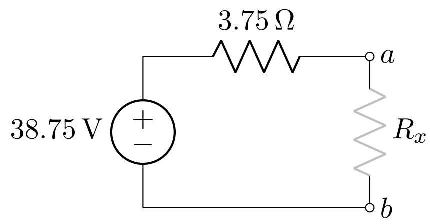
Answer 8

For this type of question, it's clearly helpful to start by finding a two-terminal equivalent (Thevenin or Norton) seen at terminals a-b, without R_x present.

There are no dependent sources in the circuit, so it's likely that the quickest way to find the source resistance is to set the independent sources to zero and calculate the equivalent resistance between 'a' and 'b'. After setting the sources to zero (remember the correct choice of open or short-circuit!) the circuit becomes a parallel $5\,\Omega$ and $15\,\Omega$ resistance, so $R_{\mathrm{T}} = 3.75\,\Omega$.

The source resistance by itself is enough to find the load resistance R_x that gives maximum power transfer. However, we also must find the Thevenin voltage (or Norton current) in order to calculate the value of the maximum power, or to choose a current-source I_x or voltage-source U_x that can replace R_x and still extract maximum power from the circuit. A convenient way to find the Thevenin voltage in this circuit is to do nodal analysis in just the node above the current source, choosing the bottom node as earth; then voltage division can be used between the $5\,\Omega$ resistors to find the open-circuit voltage. Another way — probably easier — is superposition ... try it! The Thevenin voltage is 38.75 V; the Norton current is 10.33 A.

Here is R_x connected to the Thevenin equivalent of the circuit 'behind' a-b:



i) For maximum power, $R_x = 3.75\,\Omega$.
This, of course, is from the maximum-power condition that $R_x = R_{\mathrm{T}}$.

ii) The maximum possible power from the circuit is $P_{R_x} = 100\,\mathrm{W}$ (rounded to 3 digits).
One method is to find the current from the Thevenin source with R_x connected, then to use the relation $P = i^2 R$ for the power dissipated in a resistor R by current i, to give $P = \left(\frac{U_{\mathrm{T}}}{R_{\mathrm{T}}+R_x}\right)^2 R_x$.
As we know that $R_x = R_{\mathrm{T}}$, this expression can be simplified to $\frac{U_{\mathrm{T}}^2}{4R_{\mathrm{T}}}$.

iii) With R_x set to $1\,\Omega$ and $10\,\Omega$, the power delivered by the circuit at the left of the terminals is $P_{1\,\Omega} = 66.6\,\mathrm{W}$, and $P_{10\,\Omega} = 79.4\,\mathrm{W}$.

iv) With a current source I_x connected instead of R_x, the power delivered to this source would be maximised when the source forces a current of half the Thevenin equivalent's short-circuit current. Hence, $I_x = -\frac{I_{\mathrm{N}}}{2} = -5.17\,\mathrm{A}$; the minus sign comes because the short-circuit current is downwards, which is opposite to the reference direction of the new source I_x.
The maximum power is the same as when $R_x = 3.75\,\Omega$: it's 100 W. This reminds us that the "maximum power" principle applies to the circuit behind the two terminals, regardless of what it is that we connect to it: its output is always maximised when the thing connected to it causes a current of half of the circuit's short-circuit current.

v) Similarly to part ‘iv’, the voltage source U_x should be chosen to give half of the open-circuit voltage of the circuit behind ‘a’ and ‘b’. This is another way of saying that the current from the circuit should be half of its short-circuit value (think of the straight-line relation of u and i in a Thevenin or Norton source). In this case, the +-terminal was up, so we don’t need a minus-sign; the source should be $U_x = 19.4\,\text{V}$.

Exercise 9

Consider a 230 V socket (power outlet: *eluttag*). It is commonly on a circuit rated at 10 A, using a cable with copper conductors of $1.5\,\mathrm{mm}^2$ cross-section that have about $25\,\mathrm{m\Omega/m}$ resistance for the sum of the two conductors.

Assume that the cable is 25 m long, and that its resistance is much more than the source-resistance (Thevenin resistance) of the distribution board (*elcentral*) that it connects back to. We will regard that point as a stiff source (ideal voltage source) of 230 V. (It's a good assumption that the resistance of the thick cables and supply-transformer on the other side of the distribution board is small compared to the resistance of 25 m of $1.5\,\mathrm{mm}^2$ cable.)

i) What is the Thevenin equivalent of the two-terminal circuit that an appliance (e.g. a lamp or heater) plugged into the socket will see?

ii) What is the maximum power that can be obtained from this circuit?

iii) Have you any comment on the reasonableness (or limited usefulness) of the above level of maximum power? (What current is involved in the circuit, for maximum power transfer? What voltage does the socket then supply?)

Answer 9

i) The resistance of the cable is $25\,\mathrm{m} \cdot 25\,\mathrm{m\Omega/m} = 0.625\,\Omega$. As the other resistances (at further points, the other side of the fuse-panel *elcentral*) are considered negligible, the Thevenin source resistance is just the resistance of the conductors in the cable: $R_{\mathrm{T}} = 0.625\,\Omega$. The open-circuit voltage is 230 V.

ii) The maximum power is when the current from the circuit (or its Thevenin or Norton equivalent) is half of its short-circuit value. Here, that means

$$i_{\mathrm{maxP}} = \frac{U_{\mathrm{T}}}{2R_{\mathrm{T}}} = \frac{230\,\mathrm{V}}{2 \cdot 0.625\,\Omega} = 184\,\mathrm{A}$$

The voltage across the two terminals is then half of its open-circuit value,

$$u_{\mathrm{maxP}} = \frac{U_{\mathrm{T}}}{2} = 115\,\mathrm{V}.$$

The maximum power that can be obtained from this socket is then

$$P_{\mathrm{max}} = 115\,\mathrm{V} \cdot 184\,\mathrm{A} = 21.2\,\mathrm{kW}.$$

This could alternatively have been found by the derived relation that

$$P_{\mathrm{max}} = \frac{U_{\mathrm{T}}^2}{4R_{\mathrm{T}}} = \frac{(230\,\mathrm{V})^2}{4 \cdot 0.625\,\Omega}.$$

iii) The current at the maximum-power condition sounds very high compared to the "rated current" of the circuit.

From the nice informative fuse-specifications given by IföElectric, we look in the manual for D-type fuses, and infer that an attempt at drawing 180 A would be interrupted by a 10 A fuse in less than 10 ms, i.e. within about half an ac cycle. [This is indicated by its first graph (page2) or its table (page7), and more convincingly by the graph on (page10), for reasons that we don't need to dwell on here.]

So we can conclude that these sorts of circuits are not designed to operate near a maximum-power condition! After all, if they were, then the voltage during operation would only be half of the open-circuit voltage. That wouldn't be very desirable, particularly if we have two things plugged into one socket, so one of them affects strongly the voltage of the other. We'd also be losing half the power in the wires, as heat. Instead, we aim for holding a quite stiff voltage – i.e. a low Thevenin resistance. Then the thermal limit of the wires, for carrying current without overheating, is the limitation that decides what power the circuit can supply, and what fuse to use.