Topic 06 Capacitor, Inductor, switches and steps Practice Exercises

This Topic is mainly about getting familiar with some new definitions and components.

Therefore – as with Topic 1 – there aren't really any relevant exam questions to practise on, except occasional lucky cases. Most of the related exam questions need further concepts from later Topics.

Therefore, the following exercises and the tutorials and homework are the only practice provided in the course literature. These should be sufficient: if you want more, you can find examples in books.

The rest of "transients" (Topics 7 and 8) is mainly about sudden ("step") changes happening in circuits that contain capacitors and/or inductors, as well as the other components of independent and dependent voltage and current sources, and resistors. These may be changes in the values of components (typically independent sources), or changes in the circuit topology (reconnections).

In order to make these changes, we introduce the **unit step function** and time-dependent **switches**. These are essentially simple concepts, but they may need some practice in order to get confident with them.

The **capacitor and inductor** are new components in this Topic, and they are central in all the remaining Topics. Some of the exercises are designed to reinforce the basic relations linking u and i for these components; differentiation or integration are needed. Other exercises are about handling series or parallel connection of capacitors or inductors, or calculating stored energy.

Exercise 1 Unit step: time-shift, addition, negation, etc.

Sketch (draw: *skissa*) the following time-functions.

Remember: we use the notation that $\mathbf{1}(\ )$ is the unit-step function.

a) $\mathbf{1}(t)$

b) $\mathbf{1}(2t)$

c) $\mathbf{1}(t-2)$

d) $\mathbf{1}(-t)$

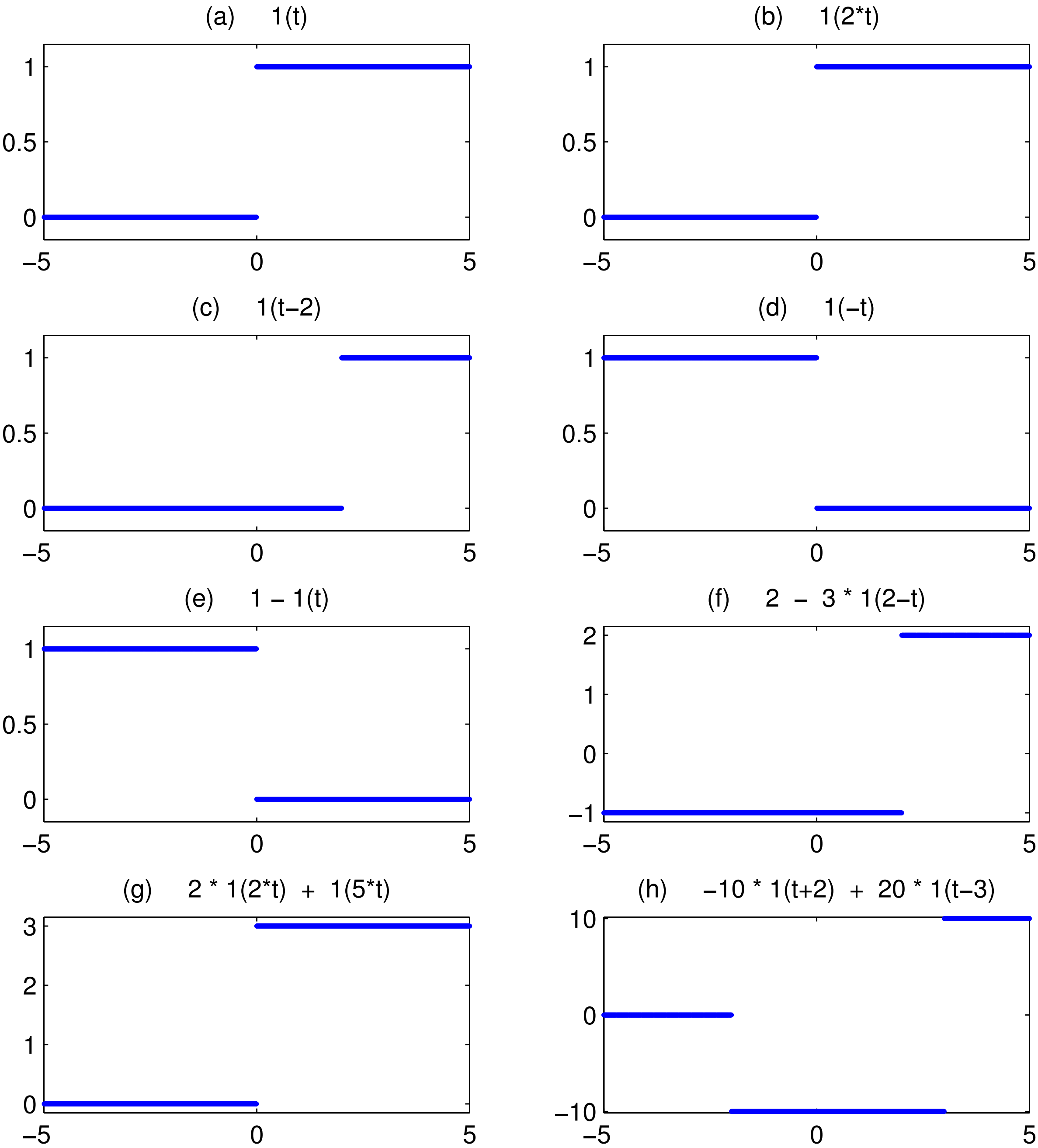
e) $1-\mathbf{1}(t)$

f) $2-3\cdot\mathbf{1}(2-t)$

g) $2\cdot\mathbf{1}(2t)+\mathbf{1}(5t)$

h) $-10\cdot\mathbf{1}(t+2)+20\cdot\mathbf{1}(t-3)$

Answer 1



Supplement to Answer 1:

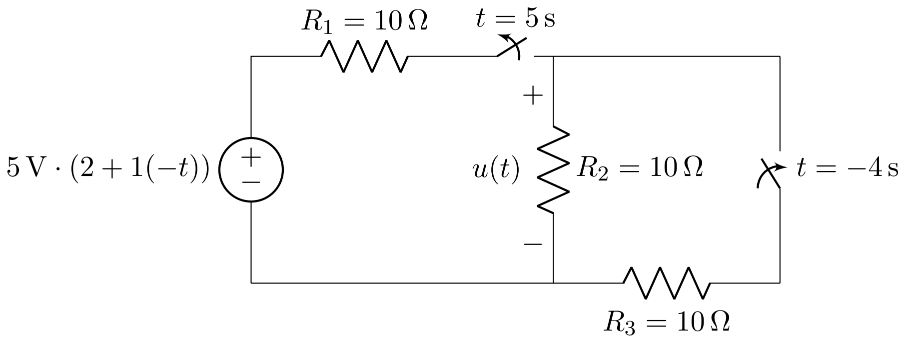
You might be interested in a Matlab example, to show how we can automate plotting the results for an arbitrary list of functions. The code that was used to produce the figure on the previous page is shown below.

On the other hand, as you're not learning Matlab just now, you're probably not as interested as other students in CENMI-åk2 who take a similar course …

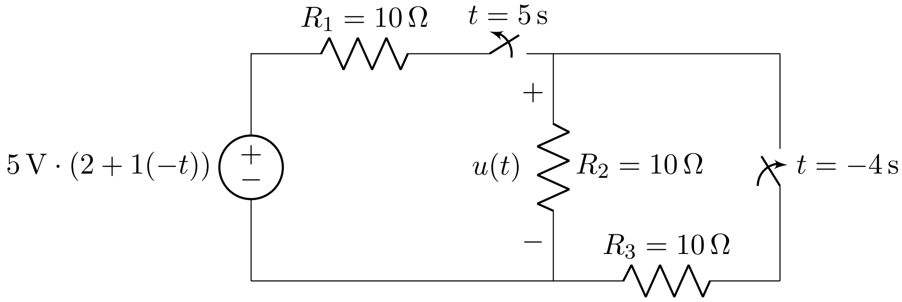
```
t = -5:0.005:5;  % a vector of time-points

ustep = @(t) double( t >= 0 );  % define a unit-step function "ustep"

% "cell-array" holding the question numbers and the equations (as text)
l = {
    'a', 'ustep(t)'
    'b', 'ustep(2*t)'
    'c', 'ustep(t-2)'
    'd', 'ustep(-t)'
    'e', '1 - ustep(t)'
    'f', '2  -  3 * ustep(2-t)'
    'g', '2 * ustep(2*t)  +  ustep(5*t)'
    'h', '-10 * ustep(t+2)  +  20 * ustep(t-3)'
};

N = size(l,1);  % number of rows in "l" (number of questions)
c = 2;  r = ceil(N/c);  % number of columns rows of plots
% for each question, plot the function in a new subplot in a single window
for n=1:N,
    subplot(r,c,n);   % set subplot number
    eval(['ft = ', l{n,2}, ';']);  % evaluate this function for all times
    plot(t, ft, '.');  % show the function in this subplot
    % set axis limits for this plot (gca is 'get current axes')
    set(gca, 'xlim', [min(t),max(t)], 'ylim', [min(ft)-0.15,max(ft)+0.15]);
    % make the subplot title out of question-number and function
    % (we change "ustep" to "1" to fit with what we use in the typesetting)
    title(sprintf('(%s)     %s', l{n,1}, regexprep(l{n,2},'ustep','1')));
end
```

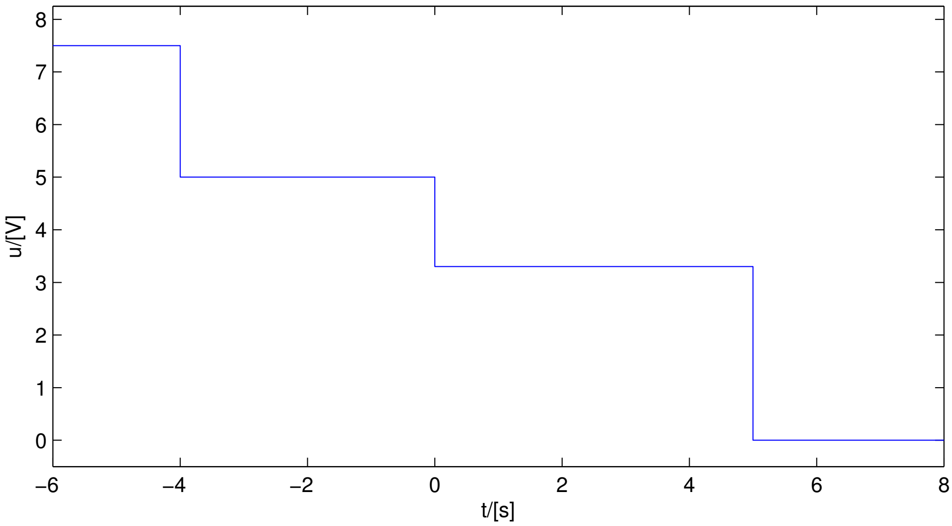
Exercise 2

Sketch a plot of $u(t)$ against time, for the following circuit.
Include at least the range $-5\,\mathrm{s} < t < 6\,\mathrm{s}$.



Answer 2

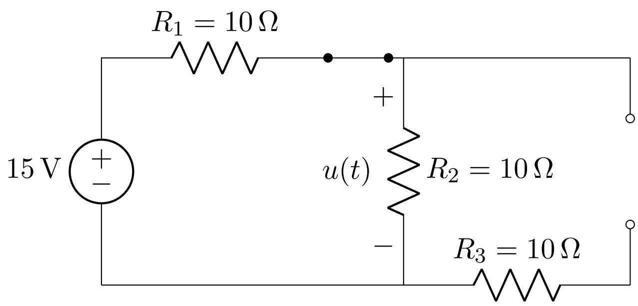




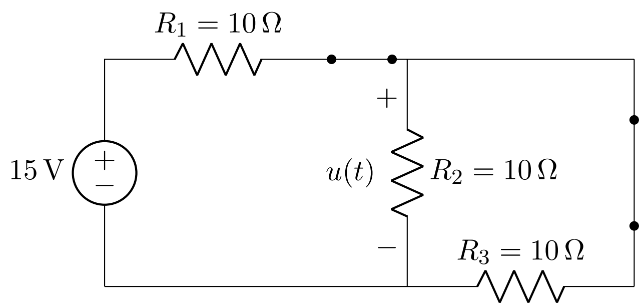
There are three changes that happen in this circuit, at times $-4\,\mathrm{s}$, $0\,\mathrm{s}$ and $5\,\mathrm{s}$. Each change is in a different component: two switches and the source.

Our most reliable plan is to consider the four periods *around* these changes: i.e., before the first change, then up to the second, then up to the third change, and after the third. In each of these periods the circuit is a simple dc circuit like the ones we've been solving before. For extra reliability we can draw each of the four situations (if we want a bigger mental challenge, we can try doing it without the drawings).

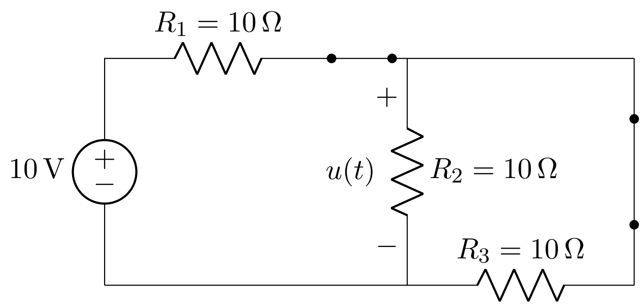
When $t < -4\,\mathrm{s}$ the left switch is closed, the right switch is open, and the source's value is $15\,\mathrm{V}$. This is shown below. By voltage division, $u = 15\,\mathrm{V}\cdot\frac{R_2}{R_1+R_2} = 7.5\,\mathrm{V}$.



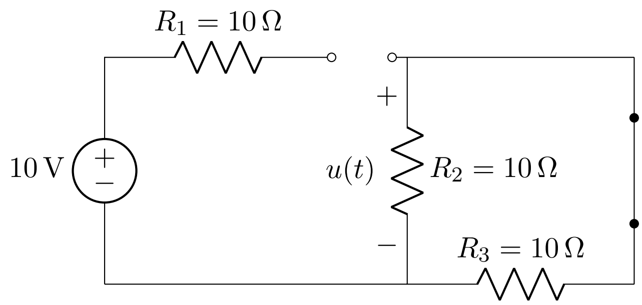
When $-4\,\text{s} < t < 0\,\text{s}$ the left switch is still closed, the right switch is now also closed, and the source's value is still $15\,\text{V}$. This is shown below. Now R_2 and R_3 are in parallel, with an equivalent value of $R_2||R_3 = \frac{R_2 R_3}{R_2+R_3} = 5\,\Omega$. By voltage division, $u = 15\,\text{V} \cdot \frac{R_2||R_3}{R_1+R_2||R_3} = 5\,\text{V}$.



When $0\,\text{s} < t < 5\,\text{s}$ the left switch is still closed, the right switch is still closed, but the source's value is now $10\,\text{V}$. The circuit has therefore the same topology as above, but with a different source-value. By voltage division, $u = 10\,\text{V} \cdot \frac{R_2||R_3}{R_1+R_2||R_3} = 3.33\,\text{V}$.



When $5\,\text{s} < t$ the left switch is open, disconnecting the only source from the rest of the circuit. The right switch is still closed, and the source's value is still $10\,\text{V}$ (but these make no difference to the fact that there is no source to drive the circuit on the right of the open switch). There is no voltage or current anywhere to the right of the open switch. Therefore, $u = 0$.
(If you want a formal proof, based on the circuit laws, then use KCL to show that the current must be equal all around the loop of R_2 and R_3, and then KVL to show that the voltage drops across the resistors, due to this current, must equal zero: assuming the resistances are positive, the solution of KVL demands that the current is zero.)

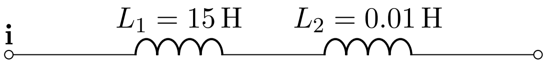


Exercise 3

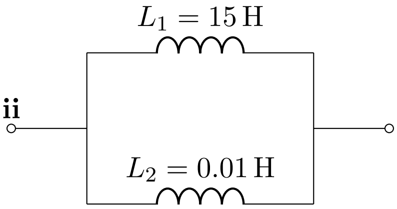
What are the equivalent values of these combinations of similar components?

Try first to answer *without* doing calculations: you can approximate (e.g. component x has much greater value than component y) or use symmetries (these have the same value, therefore ...).

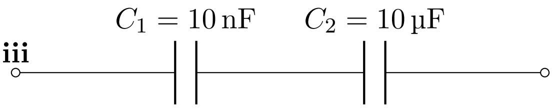
i



ii



iii



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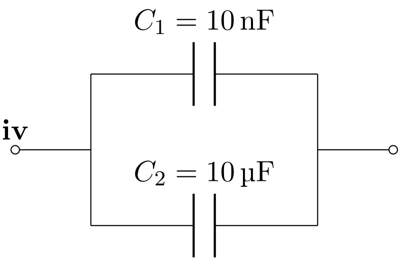
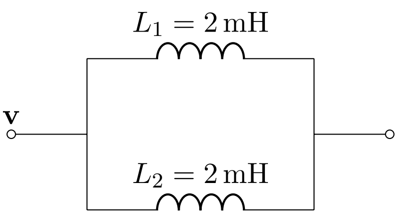
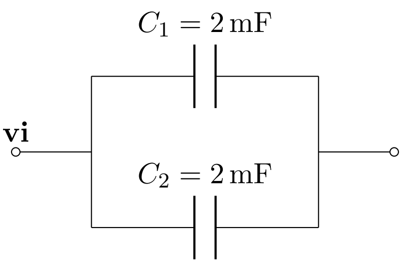


v



vi



Answer 3

i) The series 15 H and 0.01 H are approximately 15 H.
This is seen from the additive nature of series inductances, where $L_{\text{eq}} = L_1 + L_2$.
Since $L_2 \ll L_1$, $L_{\text{eq}} \simeq L_1$.

ii) The parallel 15 H and 0.01 H are approximately 0.01 H.
Parallel inductors add reciprocally, i.e. $L_{\text{eq}} = \frac{L_1 L_2}{L_1+L_2}$. The dominant one is therefore the smaller one (the one that allows more rate-of-change of current for a given voltage).

iii) The series 10 nF and 10 µF are approximately 10 nF.
Series capacitances add reciprocally, i.e. $C_{\text{eq}} = \frac{C_1 C_2}{C_1+C_2}$.
The dominant one is therefore the smaller one.

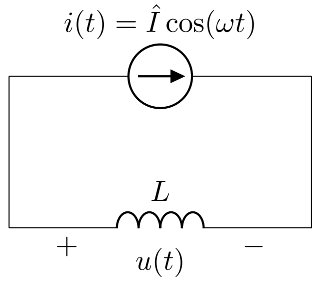
iv) The parallel 10 nF and 10 µF are approximately 10 µF.
Parallel capacitors add to give the total capacitance, $C_{\text{eq}} = C_1 + C_2$.
In this case, $C_1 \ll C_2 \implies C_{\text{eq}} \simeq C_2$. The dominant one is therefore the larger one.

v) The parallel 2 mH inductors are 1 mH.

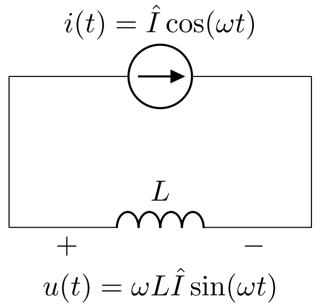
vi) The parallel 2 mF capacitors are 4 mF.

Exercise 4

Find the voltage $u(t)$, as a time-function.



Answer 4



The current source forces a current $i(t)$ in the inductor. With the definition of u and i in this circuit, the inductor's relation of u and i is

$$u(t) = -L\frac{\mathrm{d}i(t)}{\mathrm{d}t}.$$

Note the negative sign, which arises because the current i is defined into the side of the inductor where the voltage is defined as '$-$'. This is the same as with using Ohm's law in a resistor.[1]

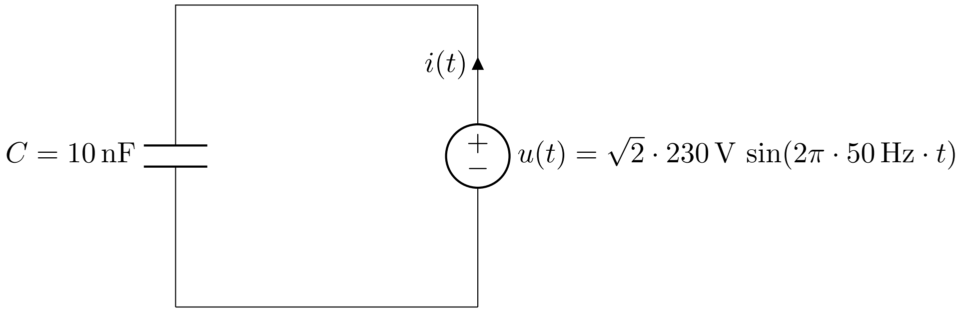
By differentiating the known time-function of current,

$$u(t) \;=\; -L\frac{\mathrm{d}}{\mathrm{d}t}\hat{I}\cos(\omega t) \;=\; -L\hat{I}\omega(-\sin(\omega t)) \;=\; \omega L\hat{I}\sin(\omega t).$$

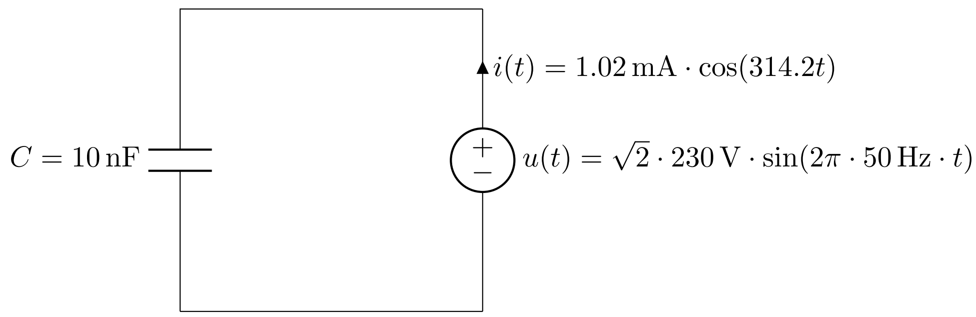
[1]One way to remember or confirm the correct relative directions is to consider that Ohm's law is about a resistor 'resisting' the flow of current by having a voltage that requires the circuit outside to do work to push current through the resistor; in an inductor, Lenz's law says that there will be a voltage to oppose the *change* in current.

Exercise 5

Find current $i(t)$.



Answer 5



The capacitor's current and voltage are related by

$$i(t) = C\frac{\mathrm{d}u(t)}{\mathrm{d}t}.$$

(This is based on the usual assumption that the current i is defined into the terminal where the voltage u has its $+$ definition. If this is not the case, a negative sign would be needed. This is just the same as when writing Ohm's law.)

In our case,

$$i(t) = \sqrt{2} \cdot 230\,\mathrm{V} \cdot 10\,\mathrm{nF} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\sin(2\pi \cdot 50\,\mathrm{Hz} \cdot t) = \sqrt{2} \cdot 230\,\mathrm{V} \cdot 2\pi \cdot 50\,\mathrm{Hz} \cdot 10\,\mathrm{nF} \cdot \cos(2\pi \cdot 50\,\mathrm{Hz} \cdot t).$$

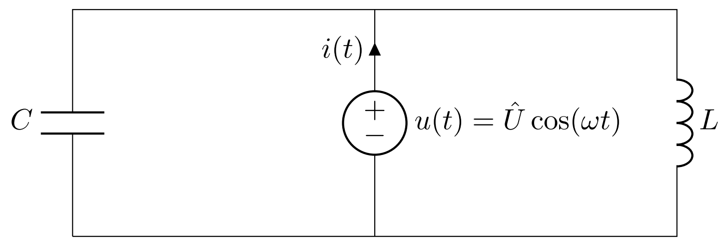
We might like to simplify the numbers to

$$i(t) = 1.02\,\mathrm{mA} \cdot \cos(314.2t).$$

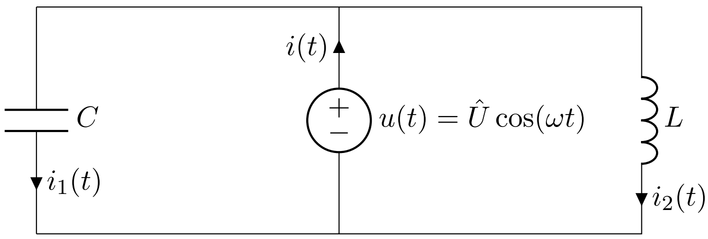
(Why the choice of numbers? The 50 Hz is a common frequency for land-based electric power systems. The standard 'low-voltage' supply in Europe is 230 V, but this describes a sinusoidal alternating current: the peak voltage of this sinusoid is $\sqrt{2}$ times the declared value. The declared value is called an 'rms' value, for reasons that should become clearer in the ac part of the course.)

Exercise 6

Find $i(t)$ in the following circuit. Assume that the current in the inductor is zero at time $t = 0$.



Answer 6



This circuit has three components in parallel. One is a voltage source, so it determines the voltage across each of the others. In this way, the inductor and capacitor are independent: each sees just a fixed voltage, regardless of what the other is doing (principle of irrelevance). It is therefore easier than it might look to find the solution.

The capacitor's u-i relation is defined as $i_1(t) = C\frac{\mathrm{d}u(t)}{\mathrm{d}t}$. Calculating this, with the given function for $u(t)$, we get

$$i_1(t) = C\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -\hat{U}\omega C\sin(\omega t).$$

The inductor's relation is $u(t) = L\frac{\mathrm{d}i_2(t)}{\mathrm{d}t}$. That's a pity. We wanted to find current, from a known voltage, not the other way around. Of course, we can re-arrange the equation, but it then becomes an integral.[2] In our ideal case, we should, strictly, consider the integration constant (initial current).

$$i_2(t) = \frac{1}{L}\int u(t)\,\mathrm{d}t = \frac{\hat{U}}{\omega L}\sin(\omega t) + i_{\mathrm{init}}.$$

This equation defines a current that is a sinusoid plus a constant. The value of i_2 is specified at one time, in the question: $i_2(0) = 0$. This can be used to find the integration constant, i_{init}. At this given time, $t = 0$, the equation for $i_2(t)$ becomes $i_2(0) = \frac{\hat{U}}{\omega L}\sin(\omega t) + i_{\mathrm{init}}$, and because $\sin(0) = 0$, this implies $0 + i_{\mathrm{init}} = 0$.

The total current $i(t)$ can then be written as

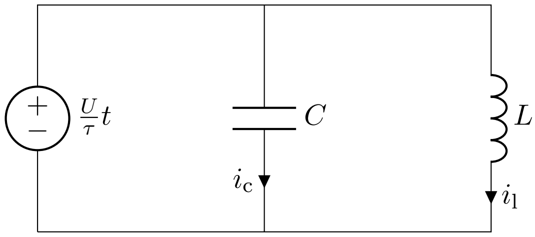
$$i(t) \;=\; i_1(t) + i_2(t) \;=\; \frac{\hat{U}}{\omega L}\sin(\omega t) - \hat{U}\omega C\sin(\omega t) \;=\; \hat{U}\left(\frac{1}{\omega L} - \omega C\right)\sin(\omega t).$$

[2] Why don't we like integrals? In general they can be more work than derivatives (needing educated guesses: but this case is simple – we can handle a cosine!); more importantly, they introduce the 'integration constant'. That's not just an annoying feature of the maths: it's an unavoidable physical feature of this situation. If an inductor has been connected to a voltage source, the current in the inductor has been *changed* by an amount that depends on the source's voltage during this time, and so the actual *value* of current depends on this change *and* current that it had at the start. In real circuits there will surely be some resistance, and any initial value of current will gradually decay to zero. (Sometimes it is nevertheless necessary to consider initial currents in inductors, or voltages on capacitors, in circuits where we care about the values shortly after making a change.)

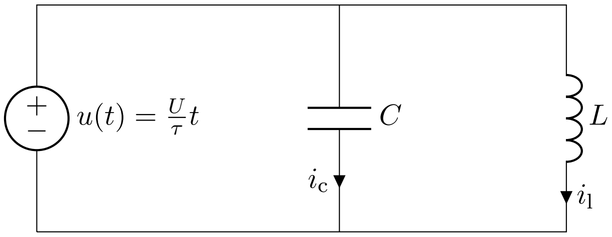
Exercise 7

Find $i_\mathrm{c}(t)$ and $i_\mathrm{l}(t)$ in the following circuit, for $t \geq 0$. Assume that $i_\mathrm{l}(0) = 0$.

(You don't need to consider what has happened *before* $t = 0$; this initial condition gives all the information that is needed about the circuit, to solve for times starting at 0.)



Answer 7



(This is basically the same as the question about a cosine voltage source with parallel C and L. The only difference is the time-function of the source, and the requirement to answer with the two currents separately.)

The time-function is a ramp: the voltage is increasing at a constant rate, given by U/τ. The initial condition of the inductor current is given. By the equations for current and voltage in a capacitor and inductor,

$$\begin{aligned} i_c(t) &= C\frac{\mathrm{d}u(t)}{\mathrm{d}t} = C\frac{\mathrm{d}}{\mathrm{d}t}\frac{U\,t}{\tau} = CU/\tau \\ i_l(t) &= \frac{1}{L}\int u(t)\mathrm{d}t = \frac{1}{L}\int \frac{U}{\tau}t\,\mathrm{d}t = \frac{U}{\tau L}\frac{t^2}{2} + i_{\text{init}}. \end{aligned}$$

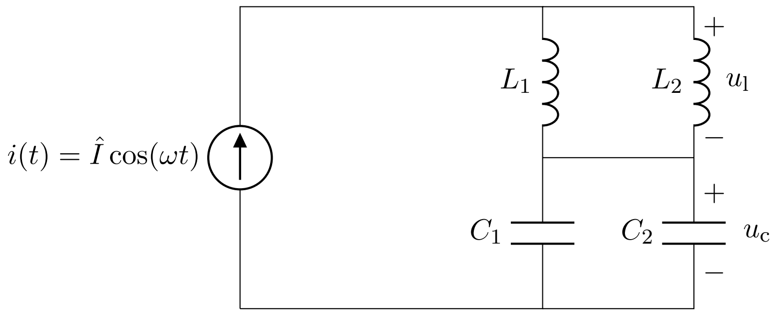
Given that $i_l(0) = 0 = \frac{U}{\tau L}\frac{0^2}{2} + i_{\text{init}}$, we see that $i_{\text{init}} = 0$. Hence,

$$\begin{aligned} i_c(t) &= CU/\tau \\ i_l(t) &= \frac{U}{\tau L}\frac{t^2}{2}. \end{aligned}$$

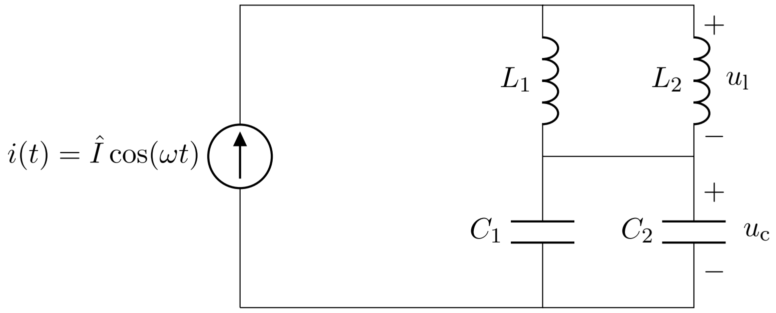
Regarding these two currents, what can be said about their shape and the difference between them? The capacitor's current is constant: the voltage rises at a constant rate, so a fixed amount of extra charge flows into the capacitor in each unit of time. The inductor's current rises at a faster and faster rate! Even a constant voltage would push the inductor's current to keep rising, so an increasing voltage makes it rise with increasing speed. This circuit could be a good model for a real circuit over some short timescale, but it is not something that could happen for a very long time! In reality, other effects, such as resistance in wires and inideality of the sources, would soon become significant, resulting in different equations.

Exercise 8

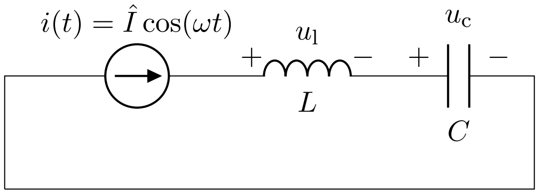
Find $u_{\mathrm{c}}(t)$ and $u_{\mathrm{l}}(t)$ in the following circuit.
Assume that at time $t = 0$, $u_{\mathrm{c}}(t) = u_0$, where u_0 is a known value.



Answer 8



This probably looks nastier than it really is. The two capacitors are in parallel: they can therefore be replaced with an equivalent, $C = C_1 + C_2$. The two inductors are also in parallel, and so can also be replaced by an equivalent, $L = \frac{L_1 L_2}{L_1+L_2}$. The circuit is then reduced to a series connection of the source $i(t)$ with an inductor L and capacitor C.



The series connection ensures that the source's current is flowing in each of the components. Thus, the voltages across the components can be calculated separately: each L or C component is irrelevant to what happens in the other one.[3] Taking the relations of current and voltage for the capacitor and inductor, the voltages can be found.

$$
\begin{aligned}
u_\mathrm{l} &= L\frac{\mathrm{d}i(t)}{\mathrm{d}t} &= L\frac{\mathrm{d}}{\mathrm{d}t}\hat{I}\cos(\omega t) &= -\hat{I}\omega L\sin(\omega t) \\
u_\mathrm{c} &= \frac{1}{C}\int \hat{I}\cos(\omega t)\mathrm{d}t &= \frac{\hat{I}}{\omega C}\sin(\omega t) + u_?
\end{aligned}
$$

Given the information that $u_\mathrm{c}(0) = u_0$, we can find the constant $u_?$, by equating

$$u_\mathrm{c}(0) = u_0 = \frac{\hat{I}}{\omega C}\sin(\omega 0) + u_? = 0 + u_?$$

from which $u_? = u_0$.

It is also necessary to replace L and C (which we defined) with the given quantities L_1 etc. After these substitutions, the answers can be given as

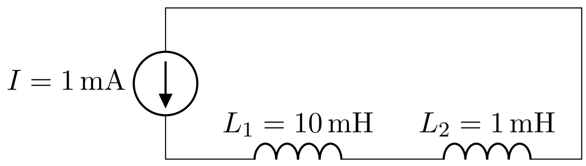
$$
\begin{aligned}
u_\mathrm{l} &= -\hat{I}\omega\frac{L_1 L_2}{L_1 + L_2}\sin(\omega t) \\
u_\mathrm{c} &= \frac{\hat{I}}{\omega(C_1 + C_2)}\sin(\omega t) + u_0
\end{aligned}
$$

(This is a 'dual' of other exercises that had parallel L and C components and a voltage source. The equations are similar, if i and u, L and C, parallel and series, etc, are swapped.)

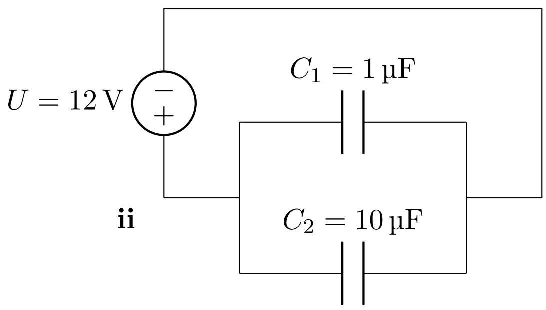
[3]This would not be true if the components were in parallel, connected to a current source, or in series, connected to a voltage source. Those cases would lead to interaction between the components' voltages or currents, leading to second-order differential equations. We're carefully avoiding such cases at this stage!

Exercise 9

For each of these four circuits, i–iv, find how many times more energy is stored in component 1 than in component 2 (this ratio might be < 1). E.g. in circuit 'i', if W_{Lx} is the energy in inductor x, then the sought ratio is $W_{\mathrm{L1}}/W_{\mathrm{L2}}$.



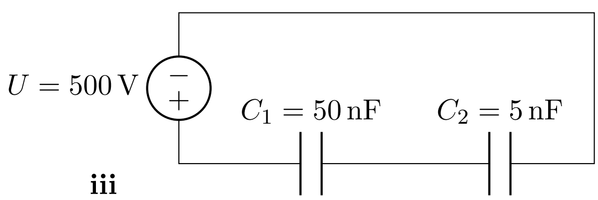
i



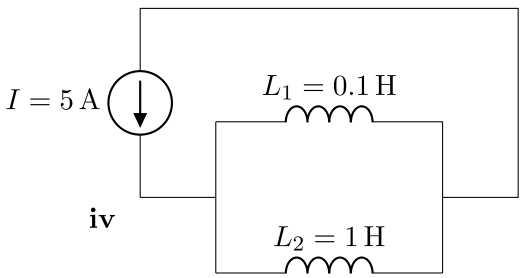
ii

In the second pair of circuits (diagrams below), it might not be so obvious what each component's voltage or current are. With similar types of component, like two inductors, you can use the same principle as voltage or current division in resistors (you might be able to derive this by considering KCL and KVL and the relations of i and u for the components). Remember that capacitances in series or parallel add like conductances, not resistances.

In both circuits, iii and iv, you can assume that the circuit started with the source set to zero, and no energy in the capacitors or inductors; then the source was increased to the stated value (does this really matter? did it matter in the previous circuits?).



iii



iv

Answer 9

The source-value (voltage or current) is not relevant to the answer in any of these circuits: only the *ratio* of energies was requested, and the energies in both components vary in the same way with the source value. For the sake of practice, we'll do some calculations of the energies in the separate components, with the given source values, as well as the requested ratios.

i) The energy in an inductor L carrying current i is $\frac{1}{2}Li^2$. The same current $I = 1\,\mathrm{mA}$ flows in both inductors, L_1 and L_2. The ratio of their energies is thus just the ratio of their inductances,

$$\frac{W_{\mathrm{L1}}}{W_{\mathrm{L2}}} = \frac{{}^1\!/\!_2 \cdot L_1 I^2}{{}^1\!/\!_2 \cdot L_2 I^2} = \frac{L_1}{L_2} = \frac{10\,\mathrm{mH}}{1\,\mathrm{mH}} = 10$$

The actual values of energy are $W_{\mathrm{L1}} = 5\,\mathrm{nJ}$ and $W_{\mathrm{L2}} = 0.5\,\mathrm{nJ}$.

ii) The energy in a capacitor C that has a voltage u is $\frac{1}{2}Cu^2$. The same voltage $U = 12\,\mathrm{V}$ is across both capacitors, C_1 and C_2. The ratio of their energies is the ratio of the capacitances.

$$\frac{W_{\mathrm{C1}}}{W_{\mathrm{C2}}} = \frac{{}^1\!/\!_2 \cdot C_1 U^2}{{}^1\!/\!_2 \cdot C_2 U^2} = \frac{C_1}{C_2} = \frac{1\,\mathrm{\mu F}}{10\,\mathrm{\mu F}} = \frac{1}{10}.$$

The actual values of energy are $W_{\mathrm{C1}} = 72\,\mathrm{\mu J}$ and $W_{\mathrm{C2}} = 720\,\mathrm{\mu J}$.

iii) This is trickier, as the voltage across each capacitor needs to be found in order to calculate the energy. Define the voltages as u_1 and u_2 on C_1 and C_2 respectively. The capacitors are in series: while they have been in this circuit, both have experienced the same current, so their charge has changed by the same amount, which we can call q.

Given that the capacitors started with no charge when the source was $0\,\mathrm{V}$ (an important piece of information), we can calculate how much charge must have passed in order to charge the series pair to the source's present value U.

$$U = u_1 + u_2 = \frac{q}{C_1} + \frac{q}{C_2} \qquad \Longrightarrow \qquad q = U\frac{C_1 C_2}{C_1 + C_2}$$

Not surprisingly, expression is quite similar to the equivalent capacitance of series-connected capacitors: we are charging a series-connected pair.

From this charge, the voltage on each capacitor can be found, and the stored energy calculated. On C_1,

$$W_{\mathrm{C1}} = \frac{1}{2}C_1 u_1^2 = \frac{1}{2}C_1\left(\frac{q}{C_1}\right)^2 = \frac{1}{2}C_1\left(\frac{1}{C_1}U\frac{C_1C_2}{C_1+C_2}\right)^2 = \frac{C_1}{2}\left(\frac{UC_2}{C_1+C_2}\right)^2.$$

The same expression is expected for W_{C2}, after swapping all the {1,2} subscripts. The ratio of energies is then

$$\frac{W_{\mathrm{C1}}}{W_{\mathrm{C2}}} = \frac{C_1}{2}\left(\frac{UC_2}{C_1+C_2}\right)^2 \Big/ \frac{C_2}{2}\left(\frac{UC_1}{C_1+C_2}\right)^2 = \frac{C_2}{C_1} = \frac{5\,\mathrm{nF}}{50\,\mathrm{nF}} = \frac{1}{10}.$$

Notice that this is the *opposite* way up compared to the earlier case: here, a smaller capacitor will store *more* energy than a bigger one, when they are in series. That's because the energy is proportional to the capacitance but also to the square of the voltage; and the voltage is inversely proportional to the capacitance.

The actual values of energy are $W_{\mathrm{C1}} = 51.7\,\mu\mathrm{J}$ and $W_{\mathrm{C1}} = 517\,\mu\mathrm{J}$.

iv) This is the dual case of 'iii)'. A difference is that we don't have a convenient name, with obvious meaning, to give to the time-integral of voltage (the time-integral of current is charge).

Again, the initial conditions were zero. Let's define the voltage across the inductors as u. The current in inductor 'x' is then given by $i_x(t_1) = {}^1/_{L_x} \int_{t_0}^{t_1} u\mathrm{d}t$, where t_0 is the time when the source started to increase from zero, and t_1 is the time when the source reached its final value I. The parallel connection of the inductors ensures that the integral of voltage is the same for both: therefore,

$$I \;=\; i_1(t_1) + i_2(t_1) \;=\; \left(\frac{1}{L_1} + \frac{1}{L_2}\right) \int_{t_0}^{t_1} u\mathrm{d}t \qquad \Longrightarrow \qquad f = I\frac{L_1 L_2}{L_1 + L_2},$$

where f is the integral of voltage, which can be eliminated later as was done with charge in the previous question.

The current in each inductor can then be found, and thereby its energy. For inductor L_1,

$$W_{\mathrm{L1}} \;=\; \frac{1}{2}L_1 i_1^2 \;=\; \frac{1}{2}L_1\left(\frac{f}{L_1}\right)^2 \;=\; \frac{1}{2}L_1\left(\frac{1}{L_1}I\frac{L_1 L_2}{L_1+L_2}\right)^2 \;=\; \frac{L_1}{2}\left(\frac{IL_2}{L_1+L_2}\right)^2.$$

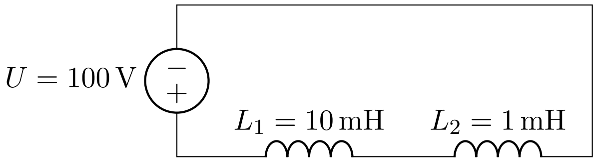
The ratio of energies is then

$$\frac{W_{\mathrm{L1}}}{W_{\mathrm{L2}}} \;=\; \frac{L_2}{L_1} \;=\; \frac{1\,\mathrm{H}}{0.1\,\mathrm{H}} \;=\; 10.$$

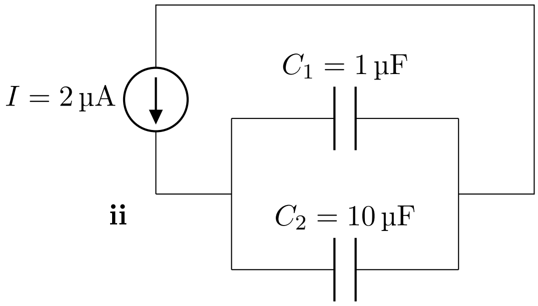
The actual values of energy are $W_{\mathrm{L1}} = 1.03\,\mathrm{J}$ and $W_{\mathrm{L1}} = 0.103\,\mathrm{J}$.

Exercise 10

What can you say about the energy stored in each of the four L or C components shown here?



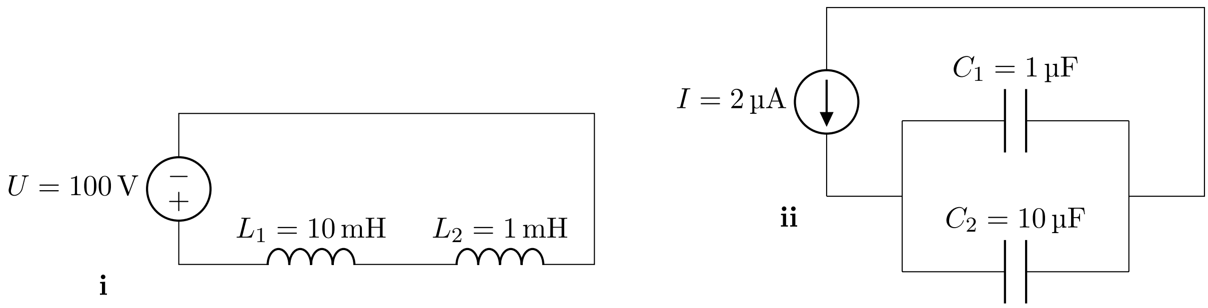
i



ii

Answer 10

"What can you say about the energy stored in each of the four L or C components?"



Answer: not much, except that after a "sufficiently long time" the energies will all be increasing, and will never stop increasing (in this idealised circuit).

Notice that the energy depends on the *continuous* variable in the component, meaning the variable that cannot have step-changes, i.e. the current in the inductor, or the voltage on the capacitor.[4] In other questions we have been able to determine the value of the continuous variable, because the sources have defined it directly: e.g. we saw a current-source connected to the inductors.

But in this question, the sources define the *other* variable, which does not directly relate to stored energy: this is the inductor voltage or capacitor current. The energy can be found by integrating this variable to find the continuous variable. This is shown, for a capacitor then for an inductor, in the following equations. We can think of the time t_0 as being when the component started to be in the circuit, at which point the continuous variable was known; then we find the continuous variable at a time t by integrating the other variable between t_0 and t.

$$\begin{aligned} u(t) &= u(t_0) + \frac{1}{C}\int_{t_0}^{t} i(x)\,\mathrm{d}x \\ i(t) &= i(t_0) + \frac{1}{L}\int_{t_0}^{t} u(x)\,\mathrm{d}x \end{aligned}$$

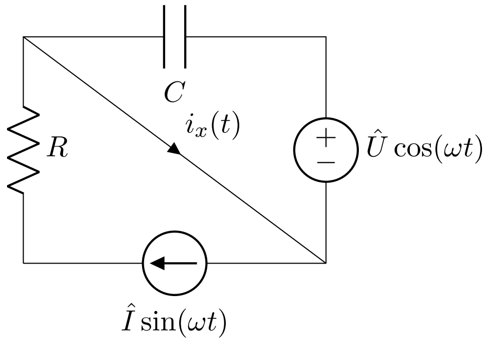
The sources in our case have constant value. Hence, the integration gives just a constant increase. Can we not then say that the energy is always increasing, even at the start? No, because we don't know the initial value of the continuous variable. For example, if a capacitor started with a very negative voltage, and the current in the circuit caused this voltage to keep increase, then the voltage would pass through zero: in that case, the stored energy would have been high at the start, then decreasing, reaching zero when the voltage is zero, and then going up again as the voltage continues rising.

What's the point of this not-very-numeric question? It is hoped to induce more thought about the continuous and non-continuous variables, and about how the difficulty of the questions depends a lot on the type of source.

[4]These variables are the continuous ones exactly *because* they are associated with stored energy: it takes time to transfer energy between the component and the circuit in order to change these variables.

Exercise 11

Determine $i_x(t)$.

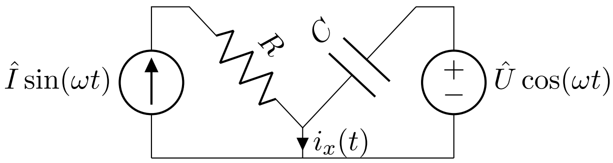


Answer 11 (this is from IT-program exam 2015-10, Q4)

The solution is

$$i_x(t) = \left(\hat{I} - \omega C \hat{U}\right) \sin(\omega t).$$

It might help to re-draw the circuit, to make clear that the two sides are independent of each other: they are two loops, with the bottom node as a common node, and the marked $i_x(t)$ is the sum of the currents in these two loops.



The two branches (R and C) can be analysed independently.

The resistor is in series with the current source, so its current is already determined as

$$i_{\mathrm{R}} = \hat{I} \sin(\omega t),$$

and the resistance R is not relevant to this current.

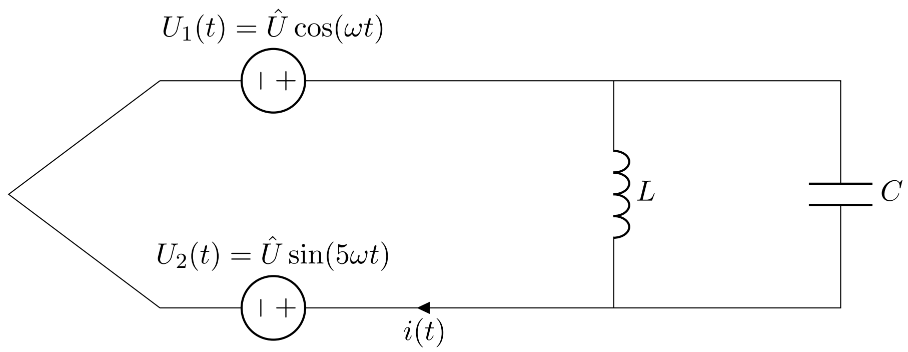
The capacitor has a voltage $\hat{U}\cos(\omega t)$ across it, which determines its current,

$$i_{\mathrm{C}} = C\frac{\mathrm{d}u}{\mathrm{d}t} = \omega C U(-\sin(\omega t)) = -\omega C U \sin(\omega t).$$

Since these two currents both are a constant multiplied by a $\sin(\omega t)$ function, their sum is easily written. This sum is the solution, as both currents return via the marked point of i_x.

Exercise 12

Find $i(t)$. Assume that at time $t = 0$ the current in the inductor is zero.



You probably feel you have not yet had enough background to be able to solve this situation, where there are *multiple sources with different frequencies.*

But you can: think superposition! It's true for linear systems, regardless of whether we're doing dc, transient or ac analysis. You can calculate the result for each source separately, then add them.

Notice also that the capacitor and inductor currents (when the same voltage is across both components, as in this parallel connection) are 180° apart, so the sine or cosine terms are easily added.

Answer 12

The capacitor and inductor are independent: KVL determines that their voltage is determined only by the voltage sources. Their currents can therefore be calculated separately and added by KCL to find $i(t)$.

To use superposition we find a result for each source, with the other set to zero (short-circuited in this case).

There are thus four calculations to consider: current in each component { inductor , capacitor } due to each source { $U_1(t)$, $U_2(t)$ } acting alone.

Source 1 active, Source 2 zeroed.

In this superposition case, the capacitor has a voltage $u(t) = U_1(t) = \hat{U}\cos(\omega t)$ at its upper node relative to its lower node. The current downwards through the capacitor is therefore

$$i_{_{\mathrm{C}(1)}}(t) = C\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -\omega C\hat{U}\sin(\omega t).$$

As this was based on a derivative (not an integral) there is no unknown constant.

The inductor has the same voltage, as it's in parallel with the capacitor.
Its current (downwards) can be expressed as

$$\frac{\mathrm{d}i_{_{\mathrm{L}(1)}}(t)}{\mathrm{d}t} = \frac{1}{L}u(t)$$

which is less easily solved, because the inductor's continuous variable (current) is dependent on the history of what the source (voltage) did to it: an integral is needed,

$$\int \frac{\mathrm{d}i_{_{\mathrm{L}(1)}}(t)}{\mathrm{d}t}\mathrm{d}t = \int \frac{1}{L}u(t)\mathrm{d}t$$

$$i_{_{\mathrm{L}(1)}}(t) = \frac{1}{L}\int u(t)\mathrm{d}t = \frac{\hat{U}}{L}\int \cos(\omega t)\mathrm{d}t = \frac{\hat{U}}{\omega L}\sin(\omega t) + k_1$$

We were given an initial condition from which the integration constant can be found: but we need to do this with both superposition cases combined, so we leave an unknown k_1 for the moment.

Source 1 active, Source 2 zeroed.

A similiar procedure to the previous case can be followed, but noting that the frequency of Source 2 is 5 times as high (substitute 5ω instead of ω) and the voltage is now defined in the opposite direction and is based on a sine instead of a cosine. The voltage $u(t)$ at top relative to bottom of the capacitor is now $u(t) = -\hat{U}\sin(5\omega t)$.

$$i_{_{\mathrm{C}(2)}}(t) = C\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -5\omega C\hat{U}\cos(5\omega t)$$

$$i_{_{\mathrm{L}(2)}}(t) = \frac{1}{L}\int u(t)\mathrm{d}t = \frac{\hat{U}}{L}\int \cos(5\omega t)\mathrm{d}t = \frac{\hat{U}}{5\omega L}\cos(5\omega t) + k_2$$

Combine the superposition states.

The sought quantity $i(t)$ is the sum of the four currents calculated above: superposition sums the currents due to the different independent sources, and KCL sums the currents of the capacitor and inductor.

$$i(t) = -\omega\hat{U}C\sin(\omega t) + \frac{\hat{U}}{\omega L}\sin(\omega t) + k_1 - 5\omega\hat{U}C\cos(5\omega t) + \frac{\hat{U}}{5\omega L}\cos(5\omega t) + k_2.$$

Let us now find the constant term, $k_1 + k_2$. The inductor current alone is

$$i_{\mathrm{L}}(t) = \frac{\hat{U}}{\omega L}\sin(\omega t) + \frac{\hat{U}}{5\omega L}\cos(5\omega t) + k_1 + k_2,$$

and we know from the given initial condition, at $t = 0$, that

$$i_{\mathrm{L}}(0) = 0 = \frac{\hat{U}}{\omega L}\sin 0 + \frac{\hat{U}}{5\omega L}\cos 0 + k_1 + k_2 = \frac{\hat{U}}{5\omega L} + k_1 + k_2, \qquad \therefore \quad k_1 + k_2 = -\frac{\hat{U}}{5\omega L}.$$

Substituting this for $k_1 + k_2$ in the expression for $i(t)$, and rearranging,

$$i(t) = \hat{U}\,\left(\left(\frac{1}{\omega L} - \omega C\right)\sin(\omega t) + \left(\frac{1}{5\omega L} - 5\omega C\right)\cos(5\omega t) - \frac{1}{5\omega L}\right).$$