KTH, Electric Circuit Analysis, EI1120

Topic 08 Time functions | Practice Exercises

Suggestion for Study:

Study the notes or chapter as a reminder of the main approaches to finding the time-function.

For an easier warm-up, you could try deriving the results found in Section 5 (pages 6&7) of the "chapter" file for this topic. These are the time-functions for different combinations of Thevenin or Norton source connected to a capacitor or inductor. Even if you've seen them derived already, it can be educational to do it independently!

There are *lots* of old exam questions that are relevant, some of which are linked below. These are the best guide to the typical level of question. You should certainly practise on plenty of these.

The exercises on the following pages in this file are not specifically easier or harder than the old exam questions. They're perhaps a bit more thoroughly described, and some are numeric. So you should choose for yourself whether you think they are useful to do also.

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2016-01˙E˙omtenta1 Q5

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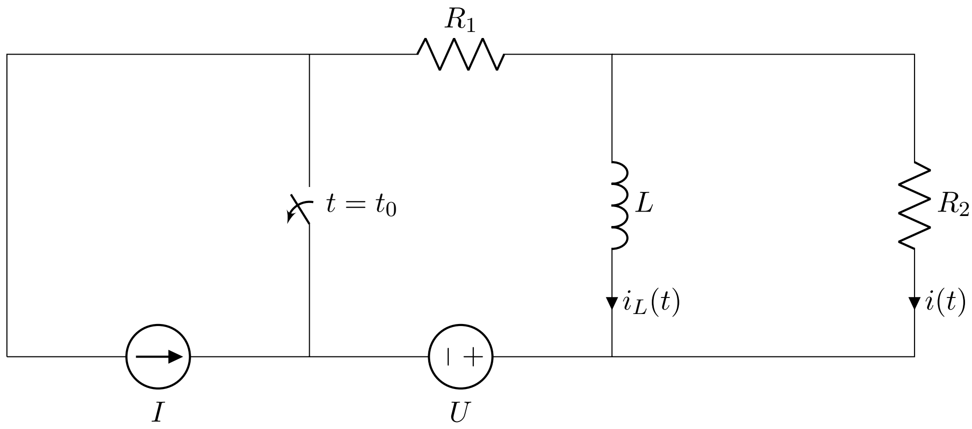
Exercise 1

Find $i(t)$ for $t > t_0$ in the following circuit.

(Assume equilibrium before t_0: this is the reasonable choice even if it's not stated, as we are not told that the circuit has any earlier changes before t_0.)

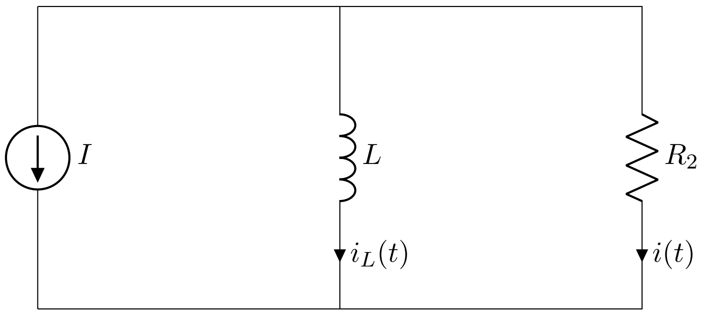
For variation, this problem has the time of the change being called $t = t_0$ instead of being defined as $t = 0$. This is to remind us that changes don't always have to happen at whatever time we've defined to be zero! We can treat the solution in just the same way, but the solution will have a time-dependence of $\mathrm{e}^{-\frac{t-t_0}{\tau}}$ instead of $\mathrm{e}^{-\frac{t}{\tau}}$.



Answer 1

At $t > t_0$, U and R_1 are irrelevant to $i(t)$ as they are in series with a current source.

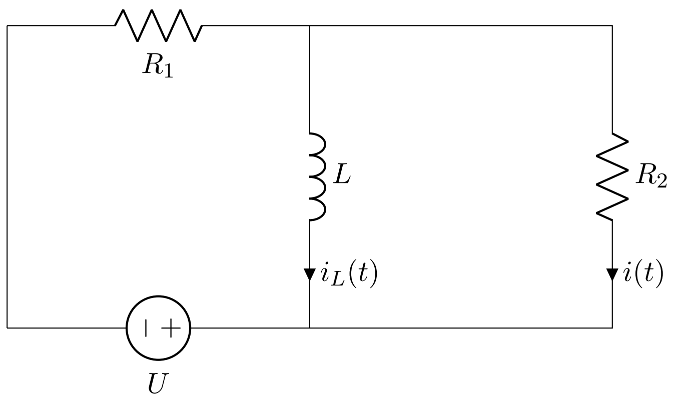
For $t > t_0$ we have:



As $t \to \infty$, we expect $i_L(\infty) = -I$ and $i(\infty) = 0$ by equilibrium.

The time constant is $\frac{L}{R_2}$ (notice that the above circuit is a Norton source with an inductor connected: $I_N = I$ and $R_N = R_2$).

Let's start by finding the continuous variable $i_L(t)$. The *initial* value, $i_L(t_0)$ cannot be found from the circuit above. We need the circuit from *before* t_0. For $t < t_0$ we have:



This, by equilibrium and continuity, shows that $i_L(t_0^+) = i_L(t_0^-) = \frac{-U}{R_1}$.

The time function $i_L(t)$ is then

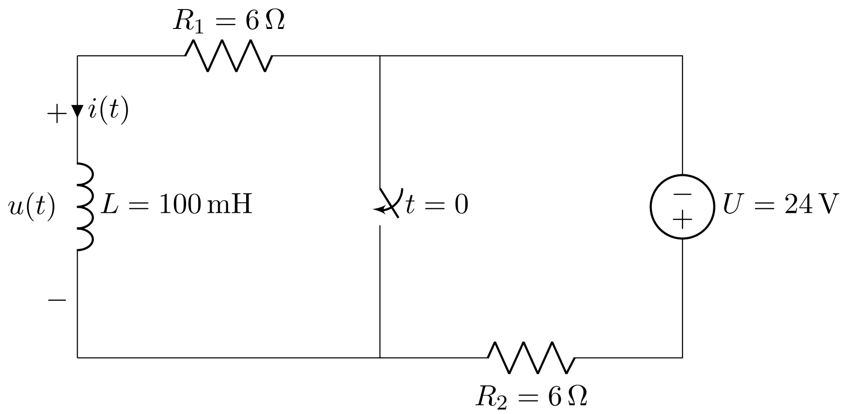
$$i_L(t) = -I + \left(\frac{-U}{R_1} + I\right) \mathrm{e}^{-\frac{(t-t_0)}{\frac{L}{R_2}}},$$

so the time function $i(t)$ is

$$i(t) = -I - i_L(t) = -I + I + \left(\frac{U}{R_1} - I\right) \mathrm{e}^{-\frac{(t-t_0)}{\frac{L}{R_2}}}.$$

Exercise 2

Find $u(t)$ and $i(t)$ in the following circuit at:
$t = 0^-$
$t = 0^+$
$t = \infty$
$t = 15\,\text{ms}$



Answer 2

At $t = 0^-$:

Equilibrium $\implies \frac{\mathrm{d}i}{\mathrm{d}t} = 0$,

$$u(t) = 0\,\mathrm{V}$$

$$i(t) = \frac{-U}{R_1 + R_2} = -2\,\mathrm{A}$$

At $t = 0^+$:

$$i(t) = -2\,\mathrm{A} \qquad \text{by continuity}$$

$$\text{by KVL, left loop} \qquad u(t) = -i(0^+)R_1 = -(-2\,\mathrm{A}) \cdot 6\,\Omega = 12\,\mathrm{V}$$

At $t = \infty$:

KVL, Ohm's law, equilibrium $\implies u(t) = 0\,\mathrm{V}$.
Equilibrium with zero voltage $\implies i(t) = 0\,\mathrm{A}$.

At $t = 15\,\mathrm{ms}$:

This time of $t = 15\,\mathrm{ms}$ is not one of the special cases of an equilibrium or 'just after' an equilibrium is disturbed. We need to consider how the quantities have changed since the known state at $t = 0^+$.

For the period $t > 0$ we know already the initial value ($t = 0^+$) and final value ($t \to \infty$) of $u(t)$ and $i(t)$.

By also determining the time-constant, we can find the complete expressions for these quantities during $t > 0$, and evaluate these at the requested time. In the circuit with the switch closed, the left loop is independent of the right loop; the inductor 'sees' a Thevenin equivalent of zero volts and with resistance of R_1. Thus,

$$\tau = \frac{L}{R_1} = \frac{100\,\mathrm{mH}}{6\,\Omega} \simeq 15\,\mathrm{ms}$$

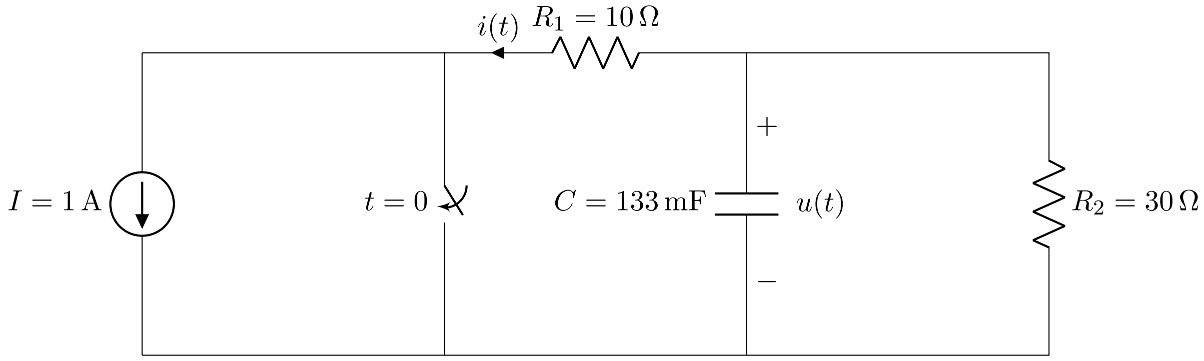
From this, $u(t) = u(0^+)\,\mathrm{e}^{-t/\tau}$ and $i(t) = i(0^+)\,\mathrm{e}^{-t/\tau}$. These are simple expressions because the final values are zero. We evaluate them with the previously found values of $u(0^+)$ and $i(0^+)$, at $t = 15\,\mathrm{ms}$. Notice that this time was chosen equal to the time-constant, so $t/\tau = 1$.

$$u(15\,\mathrm{ms}) = \mathrm{e}^{-1} \cdot 12\,\mathrm{V} \simeq 4.4\,\mathrm{V}$$

$$i(15\,\mathrm{ms}) \simeq -2\,\mathrm{A} \cdot \mathrm{e}^{-1} = -0.74\,\mathrm{A}$$

Exercise 3

Find $u(t)$ and $i(t)$ for $t > 0$ in the following circuit:



Answer 3

Equilibrium is $u(0^-) = -IR_2 = -30\,\mathrm{V}$.

Continuity $\implies u(0^+) = -30\,\mathrm{V}$.

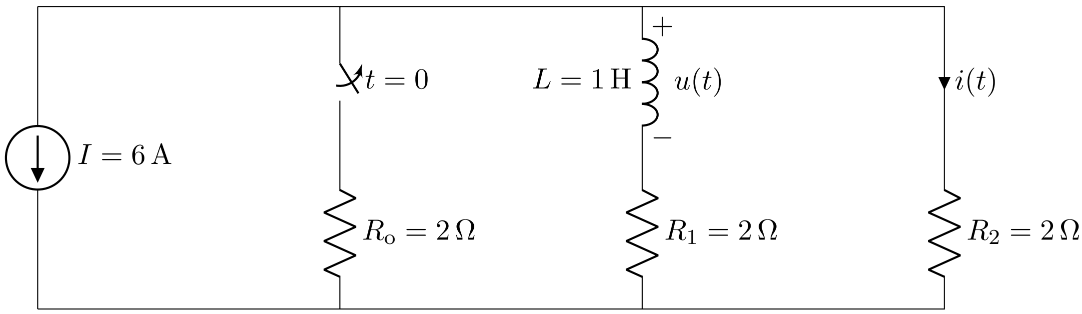
Final value $u(\infty) = 0\,\mathrm{V}$.

Time constant when $t > 0$ is $C\frac{R_1 R_2}{R_1+R_2} = 7.5\,\Omega \cdot 133\,\mathrm{mF} = 1\,\mathrm{s}$.

So: $\quad u(t) = -30\,\mathrm{V} \cdot \mathrm{e}^{\frac{-t}{1\,\mathrm{s}}}, \quad i(t) = \frac{u(t)}{10\,\Omega} = -3\,\mathrm{A} \cdot \mathrm{e}^{\frac{-t}{1\,\mathrm{s}}}$

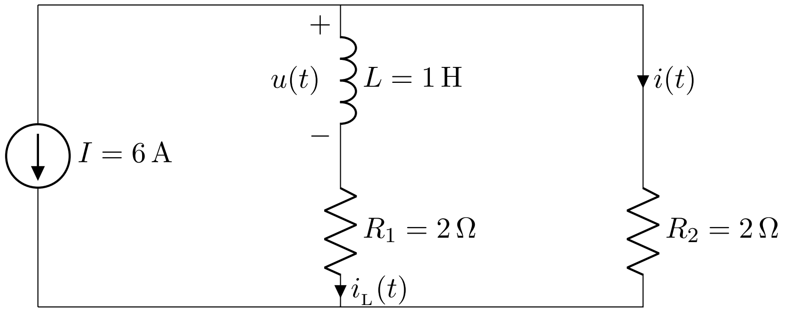
Exercise 4

Find $u(t)$ and $i(t)$ for $t > 0$ in this circuit.

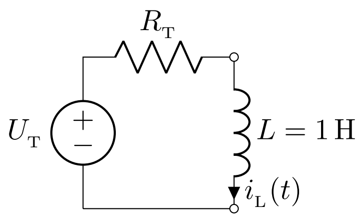


Answer 4

At $t > 0$, the circuit is:



Let's solve for the inductor's current, then find other quantities from there. Try the "intuitive" method with a Thevenin equivalent. The inductor sees a Thevenin source:



where

$$\begin{aligned} U_{\mathrm{T}} &= -IR_2 = -12\,\mathrm{V} \\ R_{\mathrm{T}} &= R_1 + R_2 = 4\,\Omega \end{aligned}$$

The time constant is $\frac{L}{R_{\mathrm{T}}} = 0.25\,\mathrm{s}$
The final value (equilibrium) of inductor current, $i_{\mathrm{L}}(\infty)$, is $\frac{U_{\mathrm{T}}}{R_{\mathrm{T}}} = -3\,\mathrm{A}$.
We can check this by current division in the original circuit!

The *initial* value cannot be found from the $t > 0$ state of the circuit or its Thevenin equivalent: it depends on what energy was stored in the inductor before the change happened, so it has to be calculated from the circuit before that change, i.e. $t = 0^-$.

At that time, current-division between *three* similar resistors gives $i_{\mathrm{L}}(0^-) = -6\,\mathrm{A} \cdot \frac{1}{3} = -2\,\mathrm{A}$.
By continuity, $i_{\mathrm{L}}(0^+) = i_{\mathrm{L}}(0^-)$.

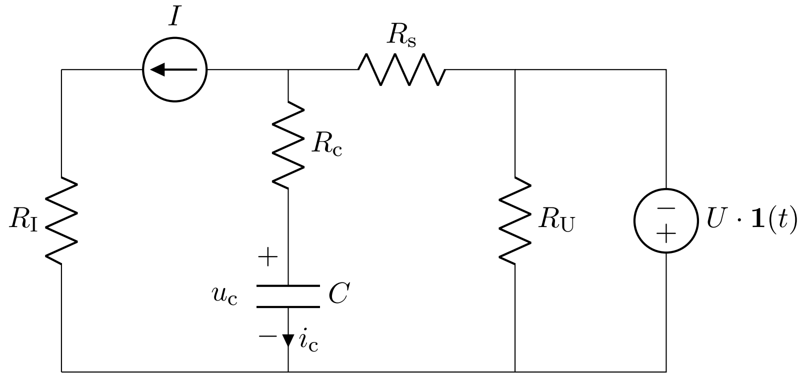
Now we know the initial value, time-constant and final value, for $i_{\mathrm{L}}(t)$.

- From the above, $i_{\mathrm{L}}(t) = -3\,\mathrm{A} + 1\,\mathrm{A} \cdot \mathrm{e}^{\frac{-t}{0.25\,\mathrm{s}}}$.
- Then $u(t) = L\frac{\mathrm{d}i_{\mathrm{L}}}{\mathrm{d}t} = \frac{-1\,\mathrm{A}\cdot 1\,\mathrm{H}}{0.25\,\mathrm{s}} \cdot \mathrm{e}^{\frac{-t}{0.25\,\mathrm{s}}} = -4\,\mathrm{V} \cdot \mathrm{e}^{\frac{-t}{0.25\,\mathrm{s}}}$,
- and $i(t) = -I - i_{\mathrm{L}}(t) = -6\,\mathrm{A} + 3\,\mathrm{A} - 1\,\mathrm{A} \cdot \mathrm{e}^{\frac{-t}{0.25\,\mathrm{s}}} = -3\,\mathrm{A} - 1\,\mathrm{A} \cdot \mathrm{e}^{\frac{-t}{0.25\,\mathrm{s}}}$.

Exercise 5

A question in two parts:
a) the familiar principle of finding a time-function after disturbing an equilibrium,
b) a *further* change, *before* a new equilibrium has been reached.

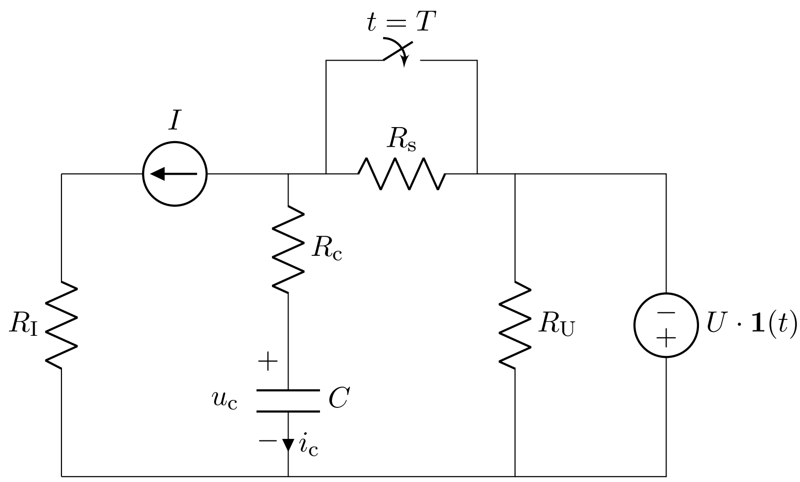
a) Find $u_c(t)$ for $t > 0$.



b) A 'Sequential switching' exercise.

This type of problem, with a change happening before equilibrium is reached, won't be asked in the exams (at least 2015, when this is written!). However, it is only a new combination of principles that you are supposed to know, so it may be useful and interesting practice.

Now we take the same circuit, but connect also a switch in parallel with R_s, closing at time $t = T$.[1]



We'll assume T is positive, i.e. this switch changes *after* the step-function of the voltage source.

The question, again, is: find $u_c(t)$ for $t > 0$.

[1]Note that this switching is equivalent to changing R_s to be $\mathbf{1}(t - T) \cdot R_s$.

This is a "sequential switching" type of problem.
First there was equilibrium ($t < 0$).
Then it was disturbed by a step function at $t = 0$.
And then a switch at $t = T$ disturbs the circuit again, *before* a new equilibrium is reached.

That is an important difference: it means we cannot use the equilibrium method (open-circuit capacitors, short-circuit inductors) to calculate the state at $t = T^{-}$, and we therefore can't use just equilibrium+continuity to find quantities at $t = T^{+}$. How can we find the initial conditions, then?

We *can* still solve the circuit for all $t > 0$. But the differential equation from part 'a)' will be needed. One reason is that it tells us $u_{\mathrm{c}}(t)$ for the period where our solution is valid: that is when $0 < t < T$. Another reason is that this equation's value at $t = T^{-}$ allows us, by continuity, to know the initial condition $u_{\mathrm{c}}(T^{+})$ just *after* the switch closes. Then we can find a new differential equation solution for the circuit at times $t > T$, after this second change in the circuit.

You have to handle the switching event by using the time-function from 'a)' to calculate the capacitor's state at time $t = T^{-}$; that is because you cannot assume equilibrium at $t = T^{-}$, since T may be a significantly long time compared to the circuit's time-constant. Then, use continuity in the usual way, to say that $u_{\mathrm{c}}(T^{+}) = u_{\mathrm{c}}(T^{-})$. Then you simply use this as a new initial value, and find the time-constant of the new circuit (after the switching), to find the time-function that is valid for $t > T$.

The final result will be the function from part 'a)', which is valid for $0 < t < T$, and the new function found here in part 'b)', valid $T < t$. As usual, we needn't bother about defining what happens exactly at $t = T$ or $t = 0$.

Answer 5

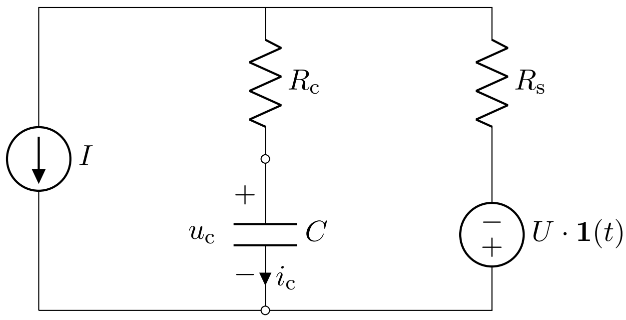
a)

The only thing about the circuit that changes is the voltage source, which is zero before $t = 0$ and constant U afterwards.

In order to find what is happening at the start of our function, $u_c(0^+)$, we need to know the voltage on the capacitor … which depends on charge … and is the continuous variable … and therefore is the same as before the change: $u_c(0^+) = u_c(0^-)$.

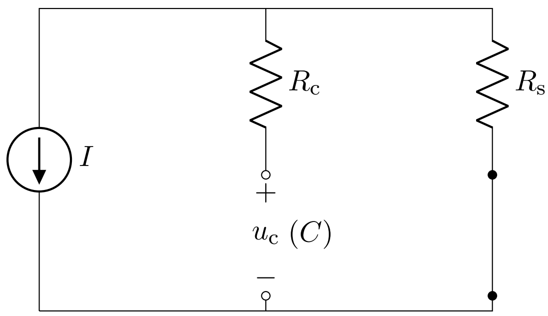
Let's start by removing the components R_{U} and R_{I}, which cannot have any effect on the capacitor, due to being "hidden" by voltage or current sources.

Then we have the following diagram, which still is a correct description of what is "seen" by the capacitor at all times (before and after 0).



At $t = 0^-$ the circuit has had constant inputs for "a very long time": this is therefore an equilibrium situation with constant values of all voltages and currents.[2]

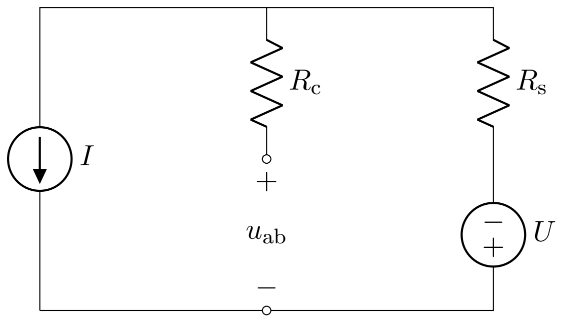
So the equilibrium voltage $u_c(0^-)$ can be found by replacing C with an open circuit, and the voltage source with a short circuit because $U \cdot \mathbf{1}(0^-) = 0$.

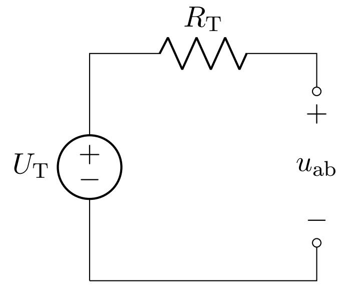


In this circuit, all the current I passes through R_{s}, so the voltage across the branch of R_{c} and C must have magnitude IR_{s}. As no current flows in C, Ohms law gives $R_{\mathrm{c}} = 0$. Therefore, taking into account the directions of definitions, $u_c(0^-) = -IR_{\mathrm{s}}$, and thus $u_c(0^+) = -IR_{\mathrm{s}}$ by continuity.

When the voltage source changes, it is clearly helpful to replace everything except the capacitor with a "two terminal equivalent".

[2] The existence of this equilibrium depends on certain conditions, e.g. not having the classic "voltage source parallel with inductor"; we can check these, but we also have a promise [this year] only to have well-behaved cases in exams.



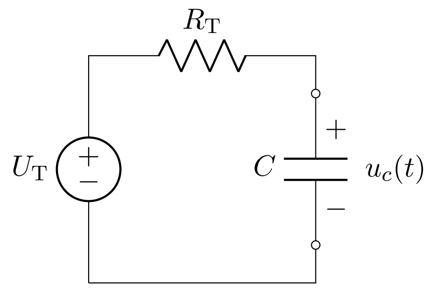


By node analysis or simply KVL in the only loop (with open-circuit instead of the capacitor), $U_T = -U - IR_s$. This can be seen from the voltage across U and R_s when no current flows in R_c. By "setting the independent sources to zero" we find that $R_T = R_c + R_s$. The more general method of finding open-circuit voltage *and* short circuit current is not needed, as we have no dependent source.

The resulting circuit is a simple series RC circuit. The differential equation is

$$RC\frac{\mathrm{d}u_c(t)}{\mathrm{d}t} + u_c(t) = U_T$$

by KVL, where $C\frac{\mathrm{d}u_c(t)}{\mathrm{d}t}$ is the current in the loop.



This has the general solution $u_c(t) = U_T + k\mathrm{e}^{-t/\tau}$, where $\tau = CR_T$.

From our earlier knowledge of the initial conditions,

$$u_c(0^+) = -IR_s = U_T + k\mathrm{e}^0 = -U - IR_s + k,$$

and therefore $k = U$. This gives the function

$$u_c(t) = -U - IR_s + U\mathrm{e}^{-t/\tau},$$

so

$$u_c(t) = -IR_s - U\left(1 - \mathrm{e}^{-t/\tau}\right).$$

b)

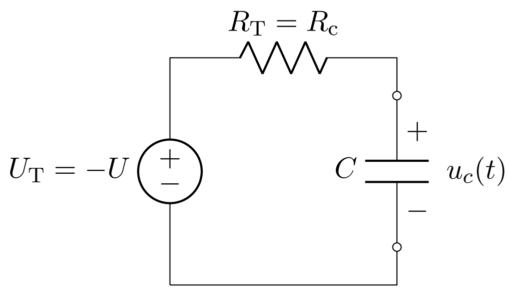
This is the same circuit as in 'a)', but now a switch shorts out R_s at $t = T$, which is equivalent to changing R_s to be $\mathbf{1}(t-T) \cdot R_s$).

Before time T there is no difference from 'a)'. So we can use the solution from 'a)' to find $u_c(t)$ in the period $0 \le t < T$.

Also — importantly — we can use that equation the capacitor's voltage at $t = T^-$. That is the continuous variable, so the value will be true also at $t = T^+$. It gives us the initial condition for a *new* equation that will be valid for the changed circuit after the switch closes ($t > T$).

From the solution to 'a)', $u_c(T^-) = -IR_\mathrm{s} - U\left(1 - \mathrm{e}^{-T/\tau}\right)$.

The new circuit is simpler when $t > T$, because the voltage source is now parallel with the other two branches: the current source therefore becomes irrelevant, and the Thevenin equivalent becomes the following.



This time let's use the idea of initial and final values and time-constant, to find the function $u_c(t)$ for $t \geq T$.

The initial value is $u_c(T^+) = -IR_\mathrm{s} - U\left(1 - \mathrm{e}^{-T/\tau}\right)$, by continuity.

The final value is $u_c(\infty) = U_\mathrm{T} = -U$.

The new time-constant is $\tau' = CR_\mathrm{c}$.

The function valid after T is therefore

$$u_c(t) = -U + \left(-IR_\mathrm{s} - U\left(1 - \mathrm{e}^{-T/\tau}\right) + U\right)\mathrm{e}^{-t/\tau'} \qquad (t > T).$$

You could even write the result as a single time-function if you really wanted, by using unit-steps to "enable" the different calculated functions in the regions where they are valid: for example,

$$f_1(t) \cdot (1 - \mathbf{1}(t-T)) \cdot \mathbf{1}(t) \;+\; f_2(t) \cdot \mathbf{1}(t-T),$$

where f_1 and f_2 are the functions valid for $0 < t < T$ and $t > T$ respectively.