KTH, Electric Circuit Analysis, EI1120

Topic 09 Practice Exercises

This is the start of 'ac'. It's another big change: this time, we don't have new components to introduce, but we have several major new ideas.

One is a *phasor* — a complex number that represents the magnitude and phase of a sinusoidal time-waveform of voltage or current (or potential or charge).

Another is *impedance* — a sort of generalised resistance, that can represent resistors, capacitors and inductors, or combinations of these; it can also be a complex number.

Consequently (see above!), it's also necessary to get familiar with *complex numbers*! We use here the symbol j to show the imaginary unit ($\sqrt{-1}$).

Please see the section on Complex Numbers, that takes up about two pages near the end of the Chapter on this topic. It gives some reminders about common ways of handling complex products, sums, etc.

The exercises here are a mixture, mainly about the skills needed for this and later topics. There is practice with complex numbers, practice with finding impedances from R, C, L values and combining into equivalent impedances, and practice with converting between time- and frequency-domain views of sinusoidal circuit quantities. Assuming you do not have time for everything, you are suggested to try exercises up to around Q5, then start looking at the old exam questions linked below.

There are some past exam questions that can be answered with just the ac skills from this topic. But note that any subquestions about calculating power are not expected to be able to be answered until next week!

2016-10˙IT˙omtenta Q5

2016-06˙IT˙omtenta Q4

2016-03˙IT˙omtenta Q4

2016-03˙E˙tenta2 Q3a

2015-10˙IT˙omtenta Q4

2015-06˙IT˙omtenta Q4

2015-03˙IT˙omtenta Q4a (only part 'a')

2014-03˙E˙tenta Q3

2015-03˙EM˙tenta Q6a

2014-08˙IT˙omtenta Q6a

2014-01˙IT˙tenta Q6a

2013-06˙EM˙omtenta Q6a

2013-03˙EM˙tenta Q6

Exercise 1

For each of the following expressions (a)–(l), express the properties (i)–(v).

The following is an example[1] for the case $z = -10\sqrt{\mathrm{j}}$.

Remember that in this subject we use j for the imaginary unit: $\mathrm{j} = \sqrt{-1}$.

i:	magnitude (absolute value)	$\lvert z \rvert = 10$
ii:	phase (argument)	$\angle z = 5\pi/4 = 225° = -3\pi/4 = -135°$
iii:	polar form ($A\angle\alpha$)	$10\angle\frac{5\pi}{4}$
iv:	rectangular form ($a + \mathrm{j}b$)	$-\frac{10}{\sqrt{2}}(1+\mathrm{j})$
v:	draw the number in the complex plane	

You can assume that $(R, L, C, \omega, \alpha)$ are all positive real numbers.

a) 1 **b)** $-\mathrm{j}$ **c)** $\mathrm{e}^{\mathrm{j}\pi}$ **d)** $\mathrm{e}^{\mathrm{j}\frac{3\pi}{2}}$ **e)** $3 + \mathrm{j}4$ **f)** $4 - 3\mathrm{j}$

g) $15\angle 135°$ **h)** $-16\angle -150°$ **i)** $A\mathrm{e}^{\mathrm{j}\alpha}$

j) $\dfrac{1}{R + \mathrm{j}\omega L}$ **k)** $R + \dfrac{1}{\mathrm{j}\omega C}$ **l)** $\dfrac{R + \mathrm{j}\omega L}{R + \mathrm{j}\omega L + R + \frac{1}{\mathrm{j}\omega C}}$

The last one is much the nastiest. Don't spend *too* much time on that.

[1]Note: in the example we're using z to denote a general complex number. That is a common convention, similar to using x for a real number: we do not mean that these complex numbers must represent impedances (which are usually upper-case, Z); they could equally well represent phasors of voltage of current.

Answer 1

The following table shows solutions for parts 'a' to 'i'.

deltal	z	$\|z\|$	$\angle z$	polar form	rectangular form
a)	1	1	0	1 (alt. $1\angle 0$)	1 (alt. 1+0j)
b)	$-\mathrm{j}$	1	$-\pi/2$	$1\angle -\frac{\pi}{2}$	0−1j
c)	$\mathrm{e}^{\mathrm{j}\pi}$	1	π	$1\angle\pi$	-1
d)	$\mathrm{e}^{\mathrm{j}\frac{3\pi}{2}}$	1	$-\pi/2$	$1\angle -\frac{\pi}{2}$	$0-1\mathrm{j}$
e)	$3+\mathrm{j}4$	5	$\tan^{-1}(4/3)$	$5\angle\tan^{-1}(4/3)$	3+j4
f)	$4-3\mathrm{j}$	5	$\tan^{-1}(-3/4)$	$5\angle\tan^{-1}(-3/4)$	4−j3
g)	$15\angle 135°$	15	135°	$15\angle\frac{3\pi}{4}$	$15\cos 135° + \mathrm{j}15\sin 135°$, alt. $\frac{15}{\sqrt{2}}(-1+\mathrm{j})$
h)	$-16\angle -150°$	16	30°	$16\angle 30°$	$16\left(\frac{\sqrt{3}}{2}+\mathrm{j}\frac{1}{2}\right)$
i)	$A\mathrm{e}^{\mathrm{j}\alpha}$	A	α	$A\angle\alpha$	$A\cos\alpha + \mathrm{j}A\sin\alpha$

Sketching an Argand diagram (a complex number drawn in the complex plane) is not shown here: it should be easy from the rectangular form, with the real and imaginary parts as the "x and y" quantities respectively.

j) $\dfrac{1}{R+\mathrm{j}\omega L}$

The denominator $R+\mathrm{j}\omega L$ has magnitude $\sqrt{R^2+(\omega L)^2}$ and angle $\tan^{-1}(\frac{\omega L}{R})$.

Thus, $\frac{1}{\text{denominator}}$ has magnitude $1/\sqrt{R^2+(\omega L)^2}$ and angle $-\tan^{-1}(\frac{\omega L}{R})$.

This gives the polar form directly.

The rectangular form can be found by multiplying the denominator and numerator by the complex conjugate of the denominator, $\frac{1}{R+\mathrm{j}\omega L} = \frac{R-\mathrm{j}\omega L}{(R+\mathrm{j}\omega L)(R-\mathrm{j}\omega L)} = \frac{R-\mathrm{j}\omega L}{R^2+\omega^2 L^2}$.

k) $R+\dfrac{1}{\mathrm{j}\omega C}$

Rectangular form: $R-\frac{\mathrm{j}}{\omega C}$.

Magnitude is $\sqrt{R^2+1/(\omega^2 C^2)}$.

Angle is $\tan^{-1}(-1/(\omega CR))$.

Polar form comes directly from the magnitude and angle.

l) $\dfrac{R+\mathrm{j}\omega L}{R+\mathrm{j}\omega L+R+1/(\mathrm{j}\omega C)}$

First, separate real and imaginary parts in the denominator,

$$R+\mathrm{j}\omega L+R+\frac{1}{\mathrm{j}\omega C} \quad\rightarrow\quad 2R+\mathrm{j}(\omega L-1/(\omega C)).$$

If only a polar solution were desired, it would be obviously best to do the whole calculation in polar form: we have a quotient t/n where $t = R + \mathrm{j}\omega L$ and $n = R + \mathrm{j}\omega L + R + \frac{1}{\mathrm{j}\omega C}$. The polar solution is

$$\frac{|t|}{|n|}\angle \underline{\angle t - \angle n},$$

which, putting in the values of t and n, becomes

$$\sqrt{\frac{R^2 + \omega^2 L^2}{4R^2 + (\omega L - 1/(\omega C))^2}}\angle\underline{\tan^{-1}\tfrac{\omega L}{R} - \tan^{-1}\tfrac{\omega L - 1/(\omega C)}{2R}}.$$

This result, in polar form, *can* be converted to rectangular form. However, we do not know actual angles, e.g. 90°, 45°, which would help us get nice values such as 1 or 0 or $\frac{1}{\sqrt{2}}$ for the rectangular components. Instead, we have just an expression for angle, in terms of several variables. In this case, the most obvious conversion to rectangular form gives a rather messy result, involving trigonometric functions.

$$\sqrt{\frac{R^2 + \omega^2 L^2}{4R^2 + (\omega L - 1/(\omega C))^2}}\cos\left(\tan^{-1}\frac{\omega L}{R} - \tan^{-1}\frac{\omega L - 1/(\omega C)}{2R}\right) + \cdots$$
$$\mathrm{j}\sqrt{\frac{R^2 + \omega^2 L^2}{4R^2 + (\omega L - 1/(\omega C))^2}}\sin\left(\tan^{-1}\frac{\omega L}{R} - \tan^{-1}\frac{\omega L - 1/(\omega C)}{2R}\right).$$

If a rectangular form is needed at a later stage of a calculation, it is therefore often best to keep everything in rectangular form, so as to avoid the above type of equation. In order to get the canonical $a + \mathrm{j}b$ rectangular form, the numerator and denominator can both be multiplied by the complex conjugate of the denominator,

$$\begin{aligned}
& \frac{R + \mathrm{j}\omega L}{2R + \mathrm{j}(\omega L - 1/(\omega C))} \cdots \\
= \quad & \frac{(R + \mathrm{j}\omega L) \quad (2R - \mathrm{j}(\omega L - 1/(\omega C)))}{(2R + \mathrm{j}(\omega L - 1/(\omega C))) \quad (2R - \mathrm{j}(\omega L - 1/(\omega C)))} \cdots \\
= \quad & \frac{\left(2R^2 + \omega^2 L^2 - \frac{L}{C}\right) + \mathrm{j}\left(2R\omega L + \frac{R}{\omega C} - R\omega L\right)}{4R^2 + \omega^2 L^2 + 1/(\omega^2 C^2) - 2L/C} \cdots \\
= \quad & \frac{2R^2 + \omega^2 L^2 - L/C}{4R^2 + \omega^2 L^2 + 1/(\omega^2 C^2) - 2L/C} + \mathrm{j}\frac{2R\omega L + \frac{R}{\omega C} - R\omega L}{4R^2 + \omega^2 L^2 + 1/(\omega^2 C^2) - 2L/C} \\
= \quad & \frac{1}{4R^2 + \omega^2 L^2 + 1/(\omega^2 C^2) - 2L/C}\left\{\left(2R^2 + \omega^2 L^2 - \frac{L}{C}\right) + \mathrm{j}\left(2R\omega L + \frac{R}{\omega C} - R\omega L\right)\right\}.
\end{aligned}$$

Exercise 2

For both the following pairs of complex numbers — a) and b) — do the following operations on the two numbers:

i: write them both in rectangular form,
multiply them in rectangular form; and
convert the result to polar form,

ii: do as in (i), but *add* instead of multiplying,

iii: swap the words 'rectangular' and 'polar' in question 'i:', then do what it says!

a) $(5 + 10\mathrm{j})$ $(-2\mathrm{e}^{-\mathrm{j}\pi/2})$

b) $10\underline{/45^\circ}$ $3(-1 + \mathrm{j})/\sqrt{2}$

Answer 2

a) $(5+10\mathrm{j})$, $(-2\mathrm{e}^{-\mathrm{j}\pi/2})$

i) Convert to rectangular form: (5+10j) , (0+2j).

Multiply in rectangular form: $(5{+}10\mathrm{j}) \cdot (2\mathrm{j}) = -20+10\mathrm{j}$.

Convert result to polar form: $\sqrt{500}\,\underline{/\pi + \tan^{-1}\frac{10}{-20}}$.

ii) Add in rectangular form: $(5{+}10\mathrm{j}) + (2\mathrm{j}) = 5+12\mathrm{j}$.

Adding in polar form "doesn't bear thinking about" unless there's a nice relation of angles.

iii) Convert to polar form: $(\sqrt{125}\,\underline{/\tan^{-1}\frac{10}{5}})$, $(2\underline{/\frac{\pi}{2}})$.

Multiply in polar form: $(\sqrt{125}\underline{/\tan^{-1}2}) \cdot (2\underline{/\frac{\pi}{2}}) = \sqrt{500}\,\underline{/(\frac{\pi}{2}+\tan^{-1}2)}$.

Convert result to rectangular form: $\sqrt{500}\cos\left(\frac{\pi}{2}+\tan^{-1}2\right) + \mathrm{j}\sqrt{500}\sin\left(\frac{\pi}{2}+\tan^{-1}2\right)$.

b) $10\underline{/45^\circ}$, $3\frac{-1+\mathrm{j}}{\sqrt{2}}$

i) Rectangular form: $\frac{10}{\sqrt{2}}(1+\mathrm{j})$, $\frac{3}{\sqrt{2}}(-1+\mathrm{j})$.

Multiply: $\frac{10}{\sqrt{2}}(1+\mathrm{j}) \cdot \frac{3}{\sqrt{2}}(-1+\mathrm{j}) = \frac{30}{2}\left\{(1+\mathrm{j})(-1+\mathrm{j})\right\} = -30$.

ii) Add: $\frac{10}{\sqrt{2}}(1+\mathrm{j}) + \frac{3}{\sqrt{2}}(-1+\mathrm{j}) = \frac{7}{\sqrt{2}} + \mathrm{j}\frac{13}{\sqrt{2}}$.

iii) Polar form: $10\underline{/\pi/4}$, $3\underline{/3\pi/4}$.

Multiply: $(10\underline{/\pi/4}) \cdot (3\underline{/3\pi/4}) = (10\cdot 3)\underline{/\frac{3\pi}{4}+\frac{\pi}{4}} = 30\underline{/\pi} = -30$.

Exercise 3

Write the phasor form (cosine reference) of the following time-domain signals:

a) $u \sin(\omega t)$

b) $\cos(\omega t) + \sin(\omega t)$

c) $u_1 \cos(\omega t + \phi_1) + u_2 \cos(\omega t - \phi_2)$

d) Why cannot $\cos(\omega t + \phi_1) + \cos(2\omega t + \phi_2)$ be converted to a phasor, as can be done for the functions in (a)–(c)?

Write the corresponding time-functions (cosine reference, peak-value) for the following phasors:

e) $A\mathrm{e}^{\mathrm{j}\alpha}$

f) $A\underline{/\alpha}$

g) $(a + \mathrm{j}b)(c + \mathrm{j}d)$

h) $\mathrm{j}\mathrm{e}^{\mathrm{j}\pi}$

i) $A\,\mathrm{e}^{\mathrm{j}+\mathrm{e}^{-\mathrm{j}\pi/2}}$

... and for these last two, which are a little different (look carefully) ...

jω) $A\mathrm{e}^{\pi/2}$

k) $A\mathrm{e}^{1+\mathrm{j}\pi}$

Answer 3

Converting between sinusoids and phasors (time and frequency 'domains')

From sinusoidal time-functions to phasors, in rectangular or polar form:

a) $u \sin(\omega t) \to u\angle -\frac{\pi}{2}$.
Because we chose (or were told by the question to choose!) a 'cosine reference', a time-function that is $\cos \omega t$ corresponds to (motsvarar) a phasor with zero angle (purely real). This mapping is an arbitrary choice. But if we're dealing with a circuit with multiple sources (all at the one frequency ω) we must choose the *same* reference for converting each of them to a phasor, in order to get the right relationship between them. And if we want to convert phasors from our solution *back* into time-domain waveforms (the sinusoidal time-functions), we must also use the same reference as we used when converting from the original time-functions to phasors; if we don't, our result will be shifted in time away from the true solution.

b) $\cos(\omega t) + \sin(\omega t) \to 1 - \mathrm{j}$.

c) $u_1 \cos(\omega t + \phi_1) + u_2 \cos(\omega t - \phi_2) \to u_1 \mathrm{e}^{\mathrm{j}\phi_1} + u_2 \mathrm{e}^{-\mathrm{j}\phi 2}$.
Because the ωt part of the two time-functions is the same, i.e. there is only one frequency in this function, it is possible to convert that function to a single phasor.
In rectangular form this phasor is $(u_1 \cos \phi_1 + u_2 \cos \phi_2) + \mathrm{j}(u_1 \sin \phi_1 - u_2 \sin \phi_2)$.
In polar form, the identities $\cos^2 \phi + \sin^2 \phi \equiv 1$, $\cos \phi \equiv \cos -\phi$, and $\sin \phi \equiv -\sin -\phi$, can be used to give

$$\sqrt{u_1^2 + u_2^2 + 2u_1u_2(\cos \phi_1 \cos \phi_2 - \sin \phi_1 \sin \phi_2)}\angle \tan^{-1} \frac{u_1 \sin \phi_1 - u_2 \sin \phi_2}{u_1 \cos \phi_1 + u_2 \cos \phi_2}.$$

d) $\cos(\omega t + \phi_1) + \cos(2\omega t + \phi_2)$. This can't be converted to a single complex number (phasor) as in the previous cases. That's because there is not just a single frequency in this function, which is a sum of a sinusoid at angular frequency ω and another at 2ω.

From phasors to sinusoidal time-functions:

e) $A\mathrm{e}^{\mathrm{j}\alpha} \to A \cos(\omega t + \alpha)$.

f) $A\angle\alpha \to A \cos(\omega t + \alpha)$.

g) $(a + \mathrm{j}b)(c + \mathrm{j}d) \to \sqrt{(a^2 + b^2)(c^2 + d^2)} \cos(\omega t + \tan^{-1} \frac{ad+cb}{ac-bd})$, provided that $ac - bd \geq 0$; otherwise, we need to use $\tan^{-1}(\frac{ad+cb}{ac-bd}) + \pi$.

h) $\mathrm{j}\mathrm{e}^{\mathrm{j}\pi} \to \cos\left(\omega t - \frac{\pi}{2}\right)$.

i) $A\mathrm{e}^{\mathrm{j}+\mathrm{e}^{-\mathrm{j}\pi/2}} \to A \cos(\omega t)$.

The following two are important reminders that we really ought to check whether the argument to $\exp(z)$ is purely imaginary before we interpret that argument as a phase angle! If there's a real part to z, then the function $\exp(z)$ is not describing just a phase but also a scaling in size, $\Re\{z\} \neq 0 \implies |\exp(z)| \neq 1$.

jω) $A\mathrm{e}^{\pi/2} \simeq A \times 4.81$. This is a real number: it has zero phase. Thus $4.81A\cos(\omega t)$.

k) $A\mathrm{e}^{1+\mathrm{j}\pi}$. Can be written $A\mathrm{e}^{1}\mathrm{e}^{\mathrm{j}\pi}$, thus $A\,\mathrm{e}\cos(\omega t+\pi)$ in time, or $-A\,\mathrm{e}\cos(\omega t)$.

Exercise 4

Impedances in series and parallel

a) A capacitor C, inductor L and resistor R are connected in *parallel*.
What is the equivalent impedance of this combination, at angular frequency ω?
Remember that you can treat impedances like resistances, but it's more work as their values can be complex.

b) What is the equivalent impedance of all three components from part 'a)' if instead they are connected *in series*?

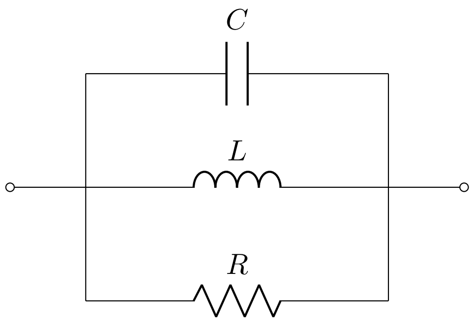
c) Express answers 'a' and 'b' in Octave/Matlab form in terms of `R, L, C, w`.
(The best way to write the imaginary unit in these languages is `1j` or `1i`. That is always an imaginary unit. The variables `i` and `j` are *initially* set to the imaginary unit during the program's startup, but they can be modified. So, to avoid getting confused at some time when you've used these as other variables, write always e.g. `X = 1j*w*L`, **not** `X = j*w*L`.)

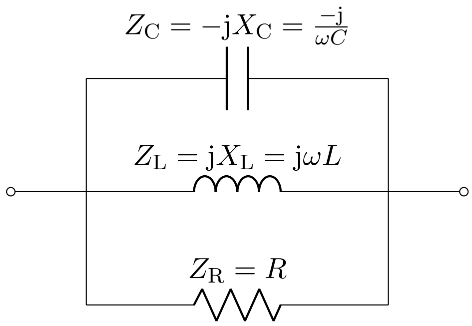
d) Find the numerical (complex) values of the expressions in 'c' if

```
R = 100, L = 5e-3, C = 2.2e-6, w = 2*pi*50.
```

Answer 4

a) Equivalent impedance of parallel connection of capacitor C, inductor L and resistor R, at angular frequncy ω.





We represent the components by their impedances, as shown on the right.
The equivalent parallel impedance Z_p between the terminals is

$$\frac{1}{Z_p} = \frac{1}{Z_R} + \frac{1}{Z_L} + \frac{1}{Z_C} = \frac{1}{R} + \frac{1}{j\omega L} + \frac{1}{-j/(\omega C)} = \frac{1}{R} + \frac{1}{j\omega L} + j\omega C,$$

and so

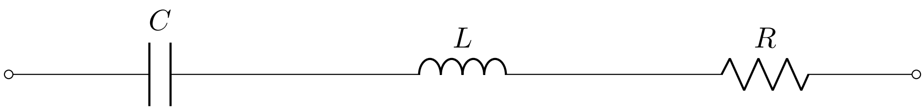
$$Z_p = \frac{1}{\frac{1}{R} + \frac{1}{j\omega L} + j\omega C} = \frac{1}{\frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)}.$$

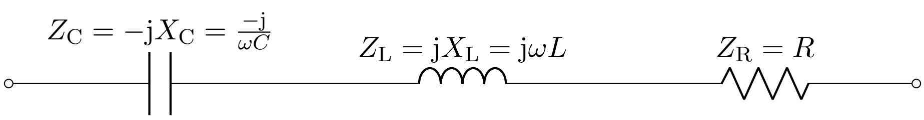
There are various ways of expressing this. The solution above is still not in $a + jb$ form. If we wanted that, we could multiply top and bottom by the complex conjugate of the part on the bottom.

However, the form shown above is quite neat and easily to "understand" (i.e. to see how Z_p changes with the component values or frequency), since each component value appears only once. So let's leave it like that, unless we come to a situation where we need to extract the real part or the angle or something like that.

For parallel connection, the *admittance*, $Y_e = 1/Z_e$, is a more suitable way to work. However, it's more common to think in terms of impedance, just as we tend to think of resistance instead of conductance.

b) Equivalent impedance for *series* connection of the same three components.





This is easier: the equivalent series impedance Z_s is

$$Z_s = Z_R + Z_L + Z_C = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right).$$

c) Express answers 'a' and 'b' in Octave/Matlab form in terms of `R, L, C, w`.

```
Zp = 1/( 1/R + 1j*( w*C - 1/(w*L) ) )
```

```
Zs = R + 1j*( w*L - 1/(w*C) )
```

Note the duality in the above!
Between series and parallel cases, just swap L and C, and R and G ($G = 1/R$).

d) Find the numerical (complex) values of the expressions in 'c' for given values:

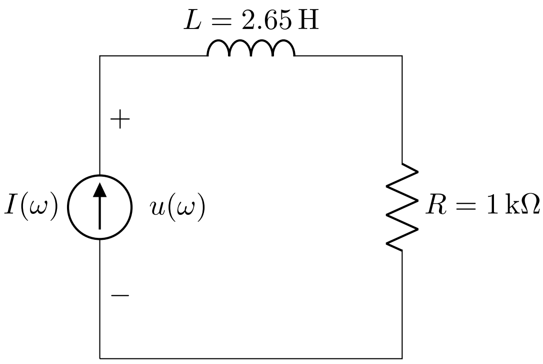
```
R = 100, L = 5e-3, C = 2.2e-6, w = 2*pi*50

Zp = 1/( 1/R + 1j*( w*C - 1/(w*L) ) )
%    0.0247+1.5721i   (~1.572 at 89.0 degrees)

Zs = R + 1j*( w*L - 1/(w*C) )
%     100.0+1445i     (~1449 at -86.0 degrees)
```

Exercise 5

Frequency, Lead-Lag.



a) What frequency (hertz) will result in a 45° difference in phase between the source current $I(\omega)$ and the voltage $u(\omega)$ marked across the source?

b) Which of these two phasor quantities, u and I, is the *lagging* one (the one that is 45° *later* than the other)?

Answer 5

a) The voltage across the current source is the product of the source current and the impedance of the external circuit,

$$u(\omega) = I(\omega) \cdot (R + \mathrm{j}\omega L) = |I(\omega)|\sqrt{R^2 + \omega^2 L^2}\,\angle\left(\angle I(\omega) + \tan^{-1}\tfrac{\omega L}{R}\right).$$

If we want to find the phase difference between this voltage and current, the actual phase of the current, $\angle I(\omega)$, is not important: the *difference* is just that the voltage leads the current by $\tan^{-1}\frac{\omega L}{R}$, i.e.

$$\angle u(\omega) - \angle I(\omega) = \angle \tfrac{u(\omega)}{I(\omega)} = \tan^{-1}\frac{\omega L}{R}.$$

This makes sense when we consider that the absolute phase of all the circuit quantities (the exact position in time of the sinusoids that they represent) is a matter of our definition of $t = 0$ and of the cosine or sine (or other) reference. It would not make sense for the *relation* of quantities in the circuit to depend on these arbitrary matters.

In order to achieve the requested 45° phase shift, we need $\tan^{-1}\frac{\omega L}{R} = 45°$.
This requires $R = \omega L$, so

$$\omega = \frac{R}{L} = \frac{1\,\mathrm{k\Omega}}{2.65\,\mathrm{H}} = 377\,\mathrm{rad/s}.$$

The required frequency is therefore

$$f = 60\,\mathrm{Hz},$$

after using the factor 2π between frequency and angular (radian) frequency.

b) The current lags the voltage by 45°.

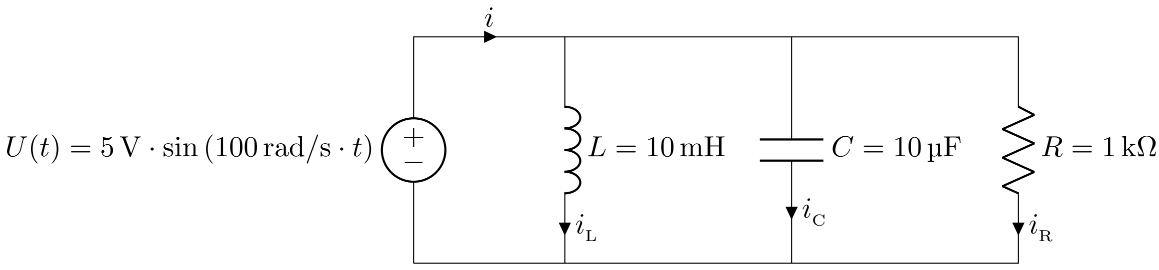
One way to justify this is that in the first expression (above) we had $u(\omega) = I(\omega) \cdot (R + \mathrm{j}\omega L)$, where R and ωL are positive quantities. Therefore, the impedance is a complex number with phase of 45° so when it multiplies the current phasor it generates a voltage phasor that *leads* the current phasor.

Exercise 6

Quickish numerical circuit-questions

a) Find the *magnitude* of current in each of the three components.

And what are the *phases* of these three currents, relative to $U(t)$?



b) What choice of angular frequency (instead of $100\,\mathrm{rad/s}$) would minimise the total current i? What is this frequency in hertz?

Answer 6

a) The circuit is driven at $\omega = 100\,\mathrm{rad/s}$.
It is convenient to use the source voltage as the phase reference, i.e. to accept a sine-reference. Thus we say that $U(\omega) = 5\,\mathrm{V}\underline{/0}$.

Then

$$i_{\mathrm{L}} = \frac{U(\omega)}{\mathrm{j}\omega L} = \frac{5\,\mathrm{V}}{100\,\mathrm{rad/s}\,\cdot\,10\,\mathrm{mH}}\underline{/-90^\circ} = -\mathrm{j}5\,\mathrm{A}.$$

From this we see the magnitude of current in the inductor is $|i_{\mathrm{L}}| = 5\,\mathrm{A}$.

Its phase is $\pi/2$ lagging the voltage. In other words, the sinusoidal time-function of current in the inductor would appear $1/4$ of a period (cycle) later than the voltage across the inductor, if seen e.g. on an oscilloscope.

We can also note that if we just wanted to find the current magnitude, we already know that this is the ratio of the voltage magnitude and impedance magnitude — then we don't have to include the angles at any part of the calculation.

Taking this quick method, for the capacitor, $|i_{\mathrm{C}}| = 5\,\mathrm{V}\cdot 100\,\mathrm{rad/s}\cdot 10\,\mu\mathrm{F}$. (This product of capacitance and angular frequency is the magnitude of *admittance* of the capacitor.) The result is $|i_{\mathrm{C}}| = 5\,\mathrm{mA}$.

Doing it more thoroughly,

$$i_{\mathrm{C}} = \frac{U(\omega)}{\frac{1}{\mathrm{j}\omega C}} = \mathrm{j}\omega C U(\omega) = 5\,\mathrm{V}\cdot 100\,\mathrm{rad/s}\cdot 10\,\mu\mathrm{F}\underline{/90^\circ} = \mathrm{j}5\,\mathrm{mA}.$$

We see that this current *leads* the source voltage in phase by 90°. The currents in the capacitor and inductor are therefore at opposite phase, 180° away from each other.

In the resistor, $i_{\mathrm{R}} = \frac{5\,\mathrm{V}}{1\,\mathrm{k}\Omega}$, i.e. $|i_{\mathrm{R}}| = 5\,\mathrm{mA}$. This is 'in phase' with the voltage: there is no phase-difference between $i_{\mathrm{R}}(\omega)$ and $U(\omega)$.

b) From the earlier question, the impedance of a parallel combination of R, L and C is

$$Z_{\mathrm{p}} = \frac{1}{\frac{1}{R} + \frac{1}{\mathrm{j}\omega L} + \mathrm{j}\omega C} = \frac{1}{\frac{1}{R} + \mathrm{j}\left(\omega C - \frac{1}{\omega L}\right)}.$$

When a voltage $U(\omega)$ is applied to this impedance, the current is

$$i(\omega) = \frac{U(\omega)}{Z_{\mathrm{p}}} = U(\omega)\,\left(\frac{1}{R} + \mathrm{j}\left(\omega C - \frac{1}{\omega L}\right)\right).$$

Notice that $\frac{1}{R}$ is purely real, whereas $\mathrm{j}\left(\omega C - \frac{1}{\omega L}\right)$ is purely imaginary. Since these parts add as a sum of squares, any reduction in magnitude of either part will reduce the total admittance, which will reduce the current. The term ω appears only in the imaginary part. To minimise the current, we must minimise this part with respect to frequency,

$$\min_{\omega}\ \left|\omega C - \frac{1}{\omega L}\right|.$$

This term is the *difference* between the admittance of the capacitor (which increases with increased frequency) and the admittance of the inductor (which decreases with frequency). We can therefore expect to find an ω that makes this term zero:

$$\omega C = \frac{1}{\omega L} \qquad \Longrightarrow \qquad \omega = \frac{1}{\sqrt{LC}}.$$

This is a special situation where the inductor current and capacitor current are equal, but (as noticed in part 'a)') 180° apart. Then their currents 'cancel' when they are driven by the same voltage.

With the given values, $\frac{1}{\sqrt{10\,\mathrm{mH}\cdot 10\,\mu\mathrm{F}}} = 3162\,\mathrm{rad/s}$, so the required frequency is about $500\,\mathrm{Hz}$.

Exercise 7

If a circuit has two sources, defined as

$$U_1(t) = A\cos(\omega t + \alpha)$$

and

$$U_2(t) = B\sin(-\omega t + \beta),$$

can these both be treated as phasors in the same calculation? In other words, are they both sinusoids at the same frequency, or can such a circuit only be solved by superposition of the results from a separate calculation for each source?

Answer 7

Negative frequencies.

Can both of these be treated as phasors in the same calculation:

$$U_1(t) = A\cos(\omega t + \alpha)$$
$$U_2(t) = B\sin(-\omega t + \beta),$$

The strange choice of $-\omega t$ in U_1 looks worrying. But this can be seen as the *same frequency* as U_2; it has sinusoidal shape, and goes through $\omega/2\pi$ cycles per unit time.

Notice that $\sin(-\theta) = -\sin(\theta) = \sin(\theta + \pi)$. (But the extra added angle β in the definition of U_2 needs to be negated, as the whole argument has been negated.)

If we decide to convert to phasors with a cosine reference and with peak values, then certainly

$$U_1(\omega) = A\angle\alpha.$$

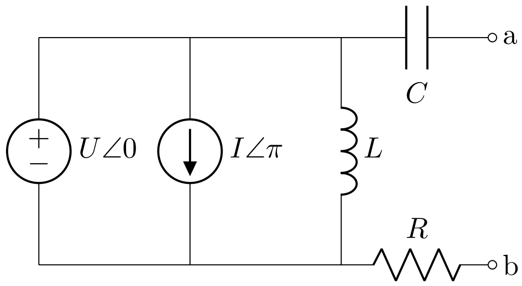
But then we can't just decide to use another reference for U_2; the function $U_2(t)$ would have to be converted to a phasors also using peak values and cosine reference,

$$U_2(\omega) = B\angle\tfrac{\pi}{2} - \beta.$$

Check that! The $\pi/2$ includes removing the negative sign from $-\omega t$, and shifting between sine and cosine.

Exercise 8

Both sources in this circuit are ac sources, with angular frequency ω. They are already expressed as complex numbers, describing the circuit in the frequency domain. It is convenient to define one source as the zero phase (the voltage source here), since a purely real phasor is more convenient for the arithmetic.



a) Find the Norton equivalent (seen from terminals a-b).
You can use the symbol Z_N for the *Norton impedance*, as the source's impedance in an ac circuit is not necessarily a pure resistance.

Warning: don't waste time on irrelevant components. Be confident about simplifications: components in an ac circuit can also be irrelevant! If in doubt, "think dc".

b) Do a dimensional analysis of your expressions for I_N and Z_N.

Note that ωC has dimension $[\Omega^{-1}]$ (admittance), and ωL is $[\Omega]$ (impedance).
Try to prove this from the general formulae $i(t) = C\frac{\mathrm{d}}{\mathrm{d}t}u(t)$ etc.
Knowing these two dimensions will be useful for dimensional analysis in later Topics.

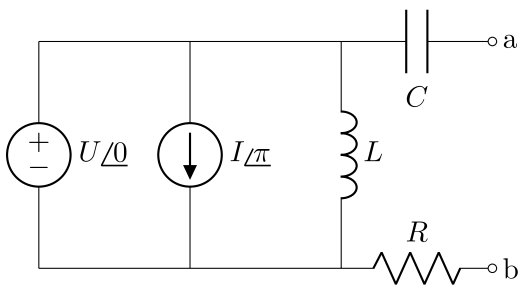
c) Do a "reasonableness check" (rimlighetskontroll) on the results from 'a'.
For example, what happens to the equivalent if $R \to \infty$ or $U \to 0$; does this make sense with the real circuit?

This is an example of a purely frequency-domain question, with no demands of converting from and to the time-domain. Voltages and currents are given as phasors. There is not even information about how these angles correspond to particular times (such as $t = 0$). That is quite typical of practical ac analysis, where one only cares about relative phase between different parts of the circuit. Your job is simply to express the solution in terms of the same reference, which can be achieved by "treat it as dc, but with complex numbers".

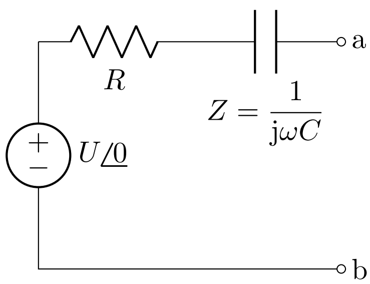
Answer 8

a) Finding the Norton equivalent between a and b.

First, irrelevance! The current-source and inductor can be removed: they are in parallel with a voltage source. Regardless of whether the solution is ac or transient or dc, they are irrelevant to the properties seen at terminals a-b. The voltage source fixes the voltage across all three of these components, by providing whatever current is needed at this voltage.



The task, then, is to find a Norton equivalent for a series connection of resistor R, voltage source $U\angle 0$, and capacitor C.



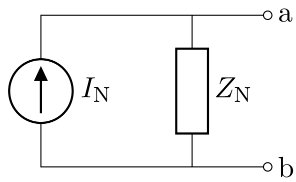
The short-circuit current between a-b is the Norton source current, which is the voltage source voltage divided by the total series impedance of the capacitor and resistor.

$$I_{\mathrm{N}} = \frac{U\angle 0}{R - \frac{\mathrm{j}}{\omega C}} = \frac{U}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}} \angle \tan^{-1}\frac{1}{\omega CR}.$$

The magic that was done in the second stage of the above equation was simply to change the denominator from rectangular to polar form, $R - \frac{\mathrm{j}}{\omega C} = \sqrt{R^2 + \frac{1}{\omega^2 C^2}} \angle -\tan^{-1}\frac{1}{\omega CR}$, and then to use the relation $\frac{1}{|z|\angle\phi} = \frac{1}{|z|}\angle -\phi$ to move the phase part into the numerator.

We could instead have expressed I_{N} in rectangular form, which is easy when the voltage source has zero phase.

The Norton impedance is also needed. We can use the rule that when there is no dependent source in a circuit, that circuit's equivalent impedance is the impedance "seen" with all its independent sources set to zero. This is the series connection of R and C,



$$Z_N = R - \frac{j}{\omega C}.$$

The Norton impedance could alternatively have been found from the open-circuit voltage and short-circuit current, $I_N = U_{oc}/I_{sc}$ where the open-circuit voltage is clearly $U_{oc} = U\underline{/0}$.

b) Dimensional check.

We've been told that ωC has dimension $[\Omega^{-1}]$. Therefore, $\frac{1}{\omega C}$ has dimension $[\Omega]$ and can correctly be added to R, and correctly equated with Z: the capacitor's reactance and the resistor's resistance are both special cases of impedance, all with the same dimension. So the expression for Z_N looks good.

In the expression for I_N the denominator has dimension $\sqrt{[\Omega]^2} = [\Omega]$, so it correctly divides into voltage U to give current I_N.

c) Reasonableness check. Just a few examples.

$R \to \infty$ causes $Z_N \to \infty$, and $I_N \to 0$.
$\omega C \to 0$ causes $Z_N \to -j\infty$ and $I_N \to j0$.

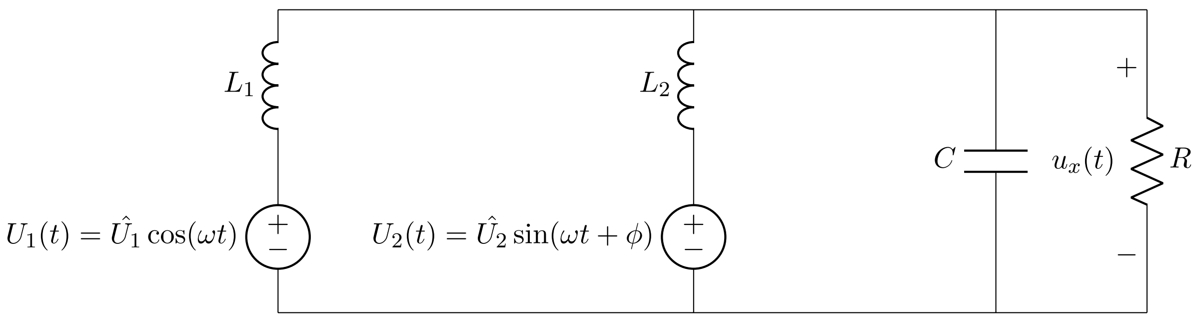
Both of the above cases increase the impedance in series with the voltage source in the original circuit. A very small capacitor, or a very large resistor, lets very little current flow. So the short-circuit current is reduced: this is seen in the reduction of the Norton source current. The Norton source impedance increases, as the open-circuit voltage must stay the same in spite of the decreased current (the original circuit has open-circuit voltage independent of the R and C component, since these have no current when there's an open-circuit).

When the capacitor is the dominant impedance, $\frac{1}{\omega C} \gg R$, the current and impedance of the Norton equivalent tend to being imaginary. This is reasonable: the voltage source has zero phase, so the current it would put into a nearly purely capacitive impedance (which is nearly purely imaginary) will also be nearly imaginary.

Exercise 9

A full ac circuit solution with multiple sources.

Find $u_x(t)$. (Assume, of course, sinusoidal steady-state solution.)



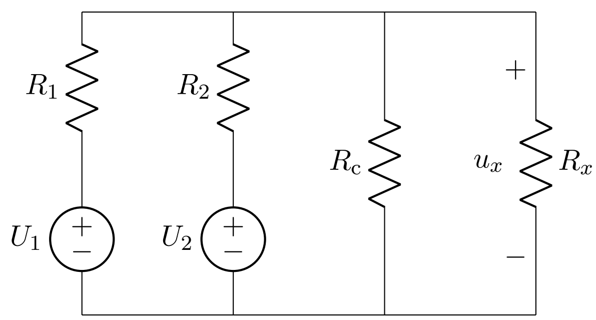
This is a classic case of time → phasors (solve) → time.
When a circuit is represented as R, L, C and time-functions, we call it a *time-domain* representation. The above circuit is shown in this way.

When a circuit is represented as impedances (complex numbers to describe R, L and C) and phasors (complex numbers to represent voltages and currents) we call it *frequency-domain* or *phasor-domain*, or say that we're using *ac analysis* or the jω-method, or various other things (just to be confusing).

The above circuit might look scary. But we will here go through all the steps needed to show that the *principle* for solving it is very simple: it ends up *looking* nasty only because we have symbolic complex expressions. Finally, we'll do it by computer (with numeric values) to show that when the algebra is done automatically, the ac calculation is almost as easy as a dc calculation in a similar circuit.

1) Both independent sources have the same frequency. They can therefore both be converted to phasors and used within a single solution: we don't *need* to use superposition.[2] When it's possible, nodal analysis is usually preferable to superposition, even as a manual solution method.

2) Before getting too worried by the circuit above, let's see how a similar sort of dc circuit would look, where all quantities are real and constant. We just replace all R, L or C with a resistor, and sources with dc sources.



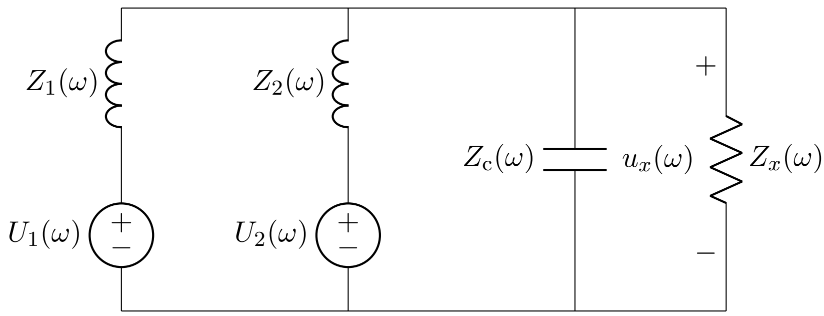
[2]We can choose to if we like that method, and we could sum the superposition results as phasors or as time-functions; in the different case where sources have different frequencies, we would have to use superposition and would have to sum the results after converting the phasors back to time-functions.

That's not too bad! The sought u_x is the difference in potential between the top and bottom nodes. Node analysis, surely: we can write a single KCL to find u_x.

- Do the nodal analysis: write KCL and rearrange to find u_x!

3) In our actual problem, of course, we have an ac circuit.
The voltage sources — like all voltages and currents — will be represented as phasors, which are complex numbers describing the magnitude and phase (angle) of sinusoidal time-functions. The impedances may be resistors, capacitors or inductors, or combinations of these: they are, in general, complex (might be real, or imaginary, or a mixture). They represent the magnitude-scaling and phase-shift between a component's current and voltage phasors.

Let us now draw the circuit again, in a way that looks more like an ac circuit!
I will even include (ω) after the phasors and impedances, to show that these can be thought of as "frequency-domain" functions. Often, we make it neater by not bothering to show that these are frequency functions; in some later Topics we'll assume everything is in the frequency-domain, and won't pollute the diagrams and equations with "(ω)" after every impedance or voltage or current!



I've chosen just to be general, by calling even the resistor "$Z_x(\omega)$", although we know that a resistor is a real, constant impedance (not frequency-dependent).

- Now you can take the equation from your dc calculation for u_x (step '2'), and replace the dc components' values (e.g. U_1, R_1) with the phasors $U_1(\omega)$ etc., and impedances $Z_1(\omega)$ etc., that are shown in the above diagram. Do that! (It's a simple step: no need to calculate anything.)

4) Finally, we have to use the known values from the original circuit to express the complex numbers for the ac calculation: phasors for the source values, and impedances for the R, L and C components.

Phasors: Voltage source 1 has a time-function of a pure $\cos(\omega t)$ (no extra phase-angle), so it seems sensible to choose a cosine reference, in order for this phasor to be purely real. If we're going to calculate everything in one go, without doing superposition, then we have to define the phasors of the independent sources with the correct *relative* phase; therefore, we should express the other source's time function as a cosine too.

- The phasor $U_1(\omega)$ is found easily as the time function already is a cosine, with phase of zero. Convert $\hat{U}_2 \sin(\omega t + \phi)$ to an equivalent cosine function. Then the phasor (complex number) representing source 2 will have magnitude of $\hat{U}_2$ and phase of whatever phase your cosine function has: the phase is the part like x in $\cos(\omega t + x)$.

Impedances: For every component marked $Z_{\text{something}}$ in the above diagram and equation, find its impedance from the values given in the first diagram, by the usual formulas such as $Z = \mathrm{j}\omega L$.

- Do that! Find all the impedances, in terms of given values L_1 etc.

5) Now that you have expressed all the phasors and impedances in terms of the known values from the first circuit, you can substitute these values into the equation you made in part '3'. Then you will have a frequency-domain circuit and equation.

- Do that, to get an equation for $u_x(\omega)$ as a function of given values (such as ω, $\hat{U}_1$, C, etc).

6) Now we have a final trouble. We need to find the time-function for $u_x(t)$.
But what we have instead is a long complex expression for $u_x(\omega)$.

We used a cosine reference for translating from time-domain to frequency-domain.
That means a phasor $A\underline{/0}$ with zero phase (purely real) corresponds to a time-function $A\cos(\omega t)$.

Thus, $u_x(t) = |u_x(\omega)| \cos\left(\omega t + \underline{/u_x(\omega)}\right)$.

So "all that we have to do" is to find u_x's magnitude and phase, $|u_x(\omega)|$ and $\underline{/u_x(\omega)}$.

That sounds easy! Let's denote the real and imaginary parts of $u_x(\omega)$ as a and b.

Then $|u_x(\omega)| = \sqrt{a^2 + b^2}$. And $\underline{/u_x(\omega))} = \tan^{-1}\frac{b}{a}$; or if $a < 0$ then $\underline{/u_x(\omega))} = \pi + \tan^{-1}\frac{b}{a}$.

Hmmmm. Actually, a and b are *not* just neat single numbers. Far from it! The expression for $u_x(\omega)$ probably occupies a whole line, and doesn't even have a distinct real and imaginary part.

This is typical of the "nasty part" when solving symbolically as phasors but then needing to go back to a time-function. In the example solution (published later) we'll also see how easy it could all be when done with numbers in Octave/Matlab. However, we need some practice at solving these small circuits by hand, even if only for the exam. In real use of circuit analysis it can also be useful to work symbolically to try to understand and prove the properties (and limitations) of a circuit.

- So here's your chance to spend some happy minutes trying to find the nicest way to express the magnitude and phase of your symbolic solution $u_x(\omega)$. Don't waste too much time. The earlier steps are the important ones for knowing *how* to do the circuit analysis. This last step is just the maths: it's often better handled by a computer, although it's probably healthy mental practice to do a bit by hand. Our exams don't tend to include such long expressions, but they do need manipulation of complex numbers, so this is good practice. Have a little go, then continue to the next question!

The **main purpose** in this question was to show the steps, and to give confidence in thinking of ac circuits as being just like dc circuits except for the need of complex numbers. I hope it has been some help! That is a good thing to remember if you feel overwhelmed by a nasty-looking ac circuit. It's also often a good principle to keep simple symbols like Z_2 or U_1 during the main part of your working, and only put in the detailed expressions like $Z_2 = R_2 + \frac{1}{\mathrm{j}\omega C_2}$ at the final stage.

Answer 9

Find the magnitude and phase of the solution, $u_x(\omega)$, then write the time-function $u_x(t)$.

What a lot of jω. We can start by dividing every term by jω.

$$u_x(\omega) = \frac{L_2\,\hat{U}_1\underline{/0} \;+\; L_1\,\hat{U}_2\underline{/\phi - \frac{\pi}{2}}}{\mathrm{j}\omega L_1 L_2\,\frac{\frac{-\mathrm{j}}{\omega C}+R_x}{\frac{-\mathrm{j}}{\omega C}R_x} + L_1 + L_2}.$$

Then handle the term that includes the capacitance: multiply every term by $\frac{-\omega C}{R}$, and simplify,

$$\frac{\frac{-\mathrm{j}}{\omega C}+R_x}{\frac{-\mathrm{j}}{\omega C}R_x} = \frac{1}{R_x} + \mathrm{j}\omega C.$$

Put this into the full equation,

$$u_x(\omega) = \frac{L_2\,\hat{U}_1\underline{/0} \;+\; L_1\,\hat{U}_2\underline{/\phi - \frac{\pi}{2}}}{\mathrm{j}\omega L_1 L_2\left(\frac{1}{R_x} + \mathrm{j}\omega C\right) + L_1 + L_2} = \frac{L_2\,\hat{U}_1\underline{/0} \;+\; L_1\,\hat{U}_2\underline{/\phi - \frac{\pi}{2}}}{L_1 + L_2 - \omega^2 L_1 L_2 C + \mathrm{j}\frac{\omega L_1 L_2}{R_x}}$$

Our aim is to find a magnitude and phase, so we can afford to convert top and bottom to polar form. The top is a *sum* of two numbers in polar form. In order to sum them, we must break them into their real and imaginary parts, for which we can use the relation $A\underline{/\alpha} = A\cos(\alpha) + \mathrm{j}A\sin(\alpha)$. Fortunately, one of the numbers is purely real ($L_2\hat{U}_1\underline{/0}$), so it doesn't need to be modified. The rectangular form for the whole numerator (top part: täljare) is

$$L_2\hat{U}_1 + L_1\hat{U}_2\cos\left(\phi - \tfrac{\pi}{2}\right) + \mathrm{j}L_1\hat{U}_2\sin\left(\phi - \tfrac{\pi}{2}\right).$$

Let us call the magnitude and angle of the numerator A_n and ϕ_n, and the magnitude and angle of the denominator (nämnare) A_d and ϕ_d. Then

$$A_n = \sqrt{\left(L_2\hat{U}_1 + L_1\hat{U}_2\cos\left(\phi - \tfrac{\pi}{2}\right)\right)^2 + \left(L_1\hat{U}_2\sin\left(\phi - \tfrac{\pi}{2}\right)\right)^2},$$

$$\phi_n = \tan^{-1}\left(\frac{L_1\hat{U}_2\sin\left(\phi - \frac{\pi}{2}\right)}{L_2\hat{U}_1 + L_1\hat{U}_2\cos\left(\phi - \frac{\pi}{2}\right)}\right)$$

and

$$A_d = \sqrt{(L_1 + L_2 - \omega^2 L_1 L_2 C)^2 + \left(\frac{\omega L_1 L_2}{R_x}\right)^2}$$

$$\phi_d = \tan^{-1}\frac{\omega L_1 L_2}{R_x\,(L_1 + L_2 - \omega^2 L_1 L_2 C)}$$

The calculation of angles by an inverse tangent assumed that the angles are in the "right half-plane" (real part non-negative); else an extra π would be needed in the phase. In the above cases, this would depend on the chosen numbers.

Now we can combine the above details, using the rule about magnitudes and phases when dividing polar complex numbers,

$$u_x(t) = \frac{A_n}{A_d}\cos\left(\omega t + \phi_n - \phi_d\right).$$

I won't even bother substituting all the expressions for A_n etc: it's not going to simplify much. If we were really trying to *use* it for something, we'd probably express it in subparts with intermediate variables (like the above), and solve it with a computer (see below).

How to do it in a few minutes by computer.

Once you've understood the concept, you can make a lot of use of computers for solving circuits. This is usually far better than using a basic calculator. It allows helpfully named intermediate variables to be used, to help keep track of the numbers at different stages of calculation. You can store all the calculations in a text file in case it later needs to be checked or a similr file is wanted in later work.

```
% choose some example data
w = 2*pi*400    % example: 400 Hz
U1p = 200,  U2p = 180,  phi = pi/4,
L1 = 1e-3,  L2 = 3e-3, C = 1e-6,  R = 40,

% find impedances and phasors
jw = 1j*w;  % to make the equations a little shorter
Z1 = jw*L1
Z2 = jw*L2
Zc = 1/(jw*C)
Zx = R
U1 = U1p  % zero phase
U2 = U2p*exp(1j*(phi-pi/2))

% let's even be too lazy to solve (rearrange) the single KCL: we
% can let matlab's symbolic toolbox do it for us
s = solve('(ux-U1)/Z1 + (ux-U2)/Z2 + ux/Zc + ux/Zx = 0', 'ux')
% s is now: (U1*Z2*Zc*Zx + U2*Z1*Zc*Zx)/(Z1*Z2*Zc + Z1*Z2*Zx + Z1*Zc*Zx + Z2*Zc*Zx)

% substitute the numeric values, and force to "double" (normal computer number);
% this step wouldn't be needed if we'd just written the equation instead of doing
% a symbolic solution of the KCL equation.
ux_w = double( subs(s) )
    1.1999e+02 + 2.6290e+01i

% find magnitude and angle
uxm = abs(ux_w)   % magnitude
    122.8
uxp = angle(ux_w)  % phase
    0.2157   % about 12.4 degrees

% make a vector of time-points, and calculate ux(t) at each
t = linspace(0, 2*2*pi/w, 1000);
ux_t = uxm * cos(w*t + uxp);

% plot the time-function
plot(t, ux_t);   xlabel('t  /s');  ylabel('u_x(t)  /V');
```

This lets us avoid all the complex algebra. Another advantage is that we don't even need to think about the little details like whether $\tan^{-1}\left(\frac{b}{a}\right)$ has a negative value of a ... the `angle()` function sorts it all out for us. Suppose that you had written a program during Part A (dc) for doing some sort of circuit calculation: e.g. nodal analysis, or finding a Thevenin equivalent. You could use that program in Octave/Matlab for ac circuits, simply by giving input data of complex numbers instead of real numbers. (Exception: power calculations, in Topic 11.) That's an example of how similar the ac and dc cases are in principle, and of what a huge simplification ac analysis is compared to solving nth-order differential equations.