**KTH, Electric Circuit Analysis, EI1120**

**Topic 12** Practice Exercises

In this topic the past exam questions provide several good exercises that are quite easily approachable if you write the equations of the components (transformer or general mutual inductance) and the rest of the circuit, and do the necessary arithmetic.

So it's suggested you **start** with the following list when you've studied the lecture/chapter material a bit. (The Exercises on the following pages of this file are not all simpler; they can be used as further practice material.)

2014-03˙E˙tenta Q2 simple mutual inductors

2014-02˙E˙ks Q3 max-power with transformer and generic $Z$ impedances

2014-03˙E˙tenta Q5 general mutual inductance, two-terminal equivalent

2013-06˙E˙omtenta Q3 fun puzzle with mutual inductance

2015-03˙EM˙tenta Q8 max-power with transformer and specific $R$ and $C$

2015-06˙IT˙omtenta Q6 quite like `2015-03_EM_tenta` Q8, but no $L$

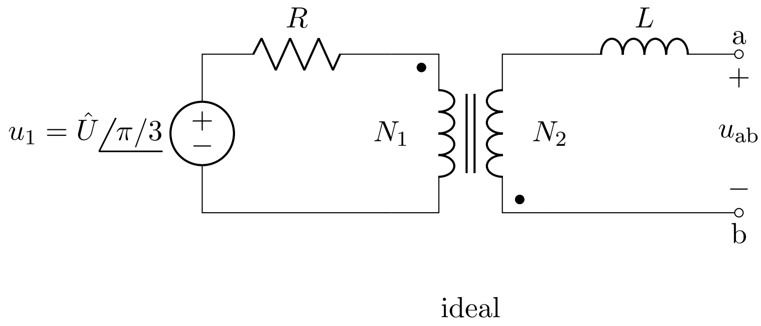
2015-06˙EM˙omtenta Q8 another variation, but using mutual inductance

2015-03˙E˙tenta2 Q6 power transfer: compare transformer and mutual inductance

2015-06˙E˙omtenta2 Q8 circuit with mutual inductance, "write but don't solve"

**Exercise 1**

Ideal transformers and Equivalent Circuit.

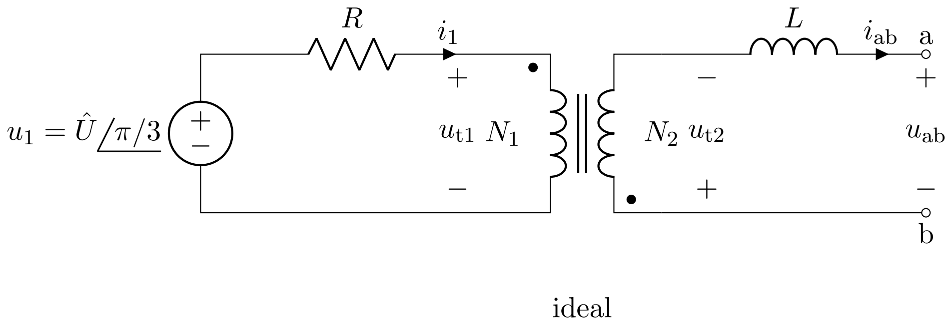



**a)** Find the Thevenin equivalent of this circuit, seen between the terminals a and b.

**b)** Do the same again, but with the inductor $L$ in *parallel* (instead of in series) with the winding $N_2$ and the output a-b.

**Answer 1**

Ideal transformers and Equivalent Circuit.




**a)** Find the Thevenin equivalent of the above circuit, seen between the terminals a and b.

1) A VERBAL STEP-BY-STEP APPROACH.

We'll start by finding the open-circuit voltage, which is the Thevenin source voltage: $u_{ab} = U_T$.

With a-b open, there is no current in $L$: therefore $i_{ab} = 0$ and $u_{ab} = -u_{t2}$.

By the transformer current equation,

$$i_{ab} = 0 \rightarrow i_1 = 0.$$

This means that no current flows through $R$, so $u_1 = u_{t1}$. By the transformer voltage equation, noting the dots and being careful about voltage definition directions,

$$u_{t2} = \frac{N_2}{N_1} u_{t1}.$$

This leads to

$$u_{ab} = U_T = -\frac{N_2}{N_1} u_1.$$

Substituting the given $u_1 = \hat{U}\angle\pi/3$, and simplifying the angle and negative sign,

$$U_T = -\frac{N_2}{N_1}\hat{U}\angle\pi/3 = \frac{N_2}{N_1}\hat{U}\angle 4\pi/3$$

Now find the source impedance. There are no dependent sources, so we can look at the impedance between terminals a-b when the source ($u_1$) is set to zero.

By the rule for translating impedances from one side to the other side of a transformer, the resistor $R$ (which is in series with the primary of the transformer) can be replaced by a resistor

$$R' = \left(\frac{N_2}{N_1}\right)^2 R$$

in series with the secondary side, without changing how the circuit behaves at the terminals a-b.

Then the total source impedance seen at a-b is

$$Z_{\mathrm{T}} = (N_2/N_1)^2 \, R + \mathrm{j}\omega L.$$

2) A MORE EQUATION-BASED APPROACH.

$$\begin{aligned} \text{KVL (primary):} \quad & u_{\mathrm{t1}} = u_1 - i_1 R, \\ \text{KVL (secondary):} \quad & u_{\mathrm{ab}} = -u_{\mathrm{t2}} - \mathrm{j}\omega L i_{\mathrm{ab}}, \\ \text{Transformer (voltage):} \quad & u_{\mathrm{t2}} = \tfrac{N_2}{N_1} u_{\mathrm{t1}}, \\ \text{Transformer (current):} \quad & i_{\mathrm{ab}} = -\tfrac{N_1}{N_2} i_1. \end{aligned} \tag{1}$$

The above are 4 equations in 5 unknowns: $i_1$, $u_{\mathrm{t1}}$, $u_{\mathrm{t2}}$, $i_{\mathrm{ab}}$, $u_{\mathrm{ab}}$.
This undetermined nature is expected, as we haven't specified anything about what is connected to a-b.

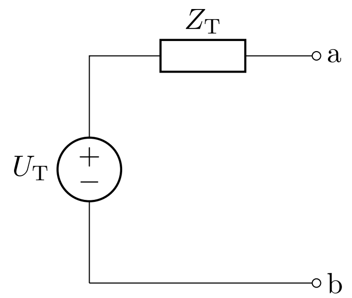
We have two popular methods available to deal with this (recall some methods from the dc topic of two-terminal equivalents).

One is to reduce the equations to a single equation relating two unknowns (terminal voltage and current), then to identify the Thevenin voltage and resistance from this equation. You can do this as an exercise if you like...

The other is to find the Thevenin equivalent from solving for two points on the circuit's $u, i$ line: we often find the short-circuit and open-circuit cases to be the most convenient pair.
To find open-circuit voltage, set $i_{\mathrm{ab}} = 0$, solve for $u_{\mathrm{ab(oc)}}$.
To find short-circuit current, set $u_{\mathrm{ab}} = 0$, solve for $i_{\mathrm{ab(sc)}}$.

Using these two results, the Thevenin equivalent is:

$$\begin{aligned} U_{\mathrm{T}} = u_{\mathrm{ab(oc)}} = \frac{-N_2}{N_1} u_1 = & \; \frac{N_2}{N_1} \hat{U} \underline{/4\pi/3}, \\ Z_{\mathrm{T}} = \frac{u_{\mathrm{ab(oc)}}}{i_{\mathrm{ab(sc)}}} = & \; \frac{N_2^2}{N_1^2} R + \mathrm{j}\omega L. \end{aligned} \tag{2}$$



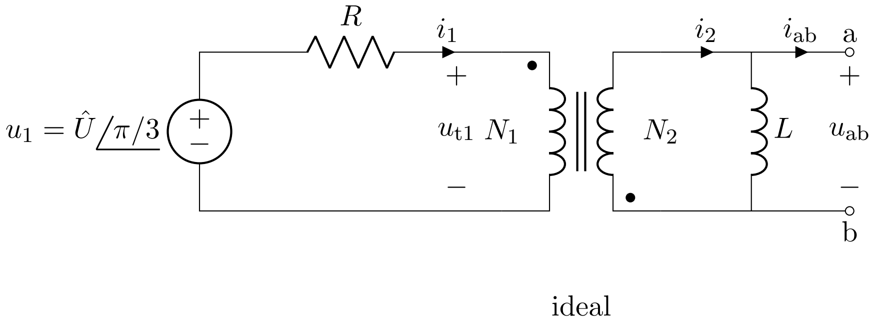

These are – fortunately – the same results as were found by the first approach.

Rather than going through a step-by-step derivation of the final equations (2) from the initial circuit equations (1), let's show how useful a symbolic-algebra computer program can be for this purpose! See the next page.

Using the symbolic-algebra program we can keep the equations in their original form (with clear relation to the circuit) and make modifications easily then recompute the result. (Yes, it's not available in the exam: this is mentioned for "real" use in later courses, projects, jobs!)

```
% the equations of the circuit
eqns = { ...
        'ut1 = u1 - i1*R', ...
        'uab = -ut2 - sqrt(-1)*w*L*iab', ...
        'ut2 = (N2/N1)*ut1', ...
        'iab = -(N1/N2)*i1'  }
unknowns = 'ut1, i1, ut2, uab, iab'

% solve for the unknowns, with the extra condition of an OPEN-circuit
soc = solve( eqns{:}, 'iab = 0', unknowns )
uaboc = soc.uab
%> uaboc = -(N2*u1)/N1

% solve for the unknowns, with the extra condition of a SHORT-circuit
ssc = solve( eqns{:}, 'uab = 0', unknowns )
iabsc = ssc.iab
%> iabsc = -(N1*N2*u1)/(L*w*N1^2*i + R*N2^2)

Ut = simplify( uaboc )
%> Ut = -(N2/N1)*u1

Zt = simplify( uaboc / iabsc )
%> Zt = R*(N2/N1)^2  +  1j*w*L
```

**b)** Do the same again, but with the inductor $L$ in *parallel* (instead of in series) with the winding $N_2$ and the output a-b.




Let's use the equation approach, and modify the equations (1) to suit the new situation,

$$\begin{aligned}
\text{KVL (primary):} \quad & u_{\mathrm{t1}} = u_1 - i_1 R, \\
\text{KCL (secondary):} \quad & i_{\mathrm{ab}} = i_2 - \frac{u_{\mathrm{ab}}}{\mathrm{j}\omega L}, \\
\text{Transformer (voltage):} \quad & u_{\mathrm{ab}} = -\tfrac{N_2}{N_1} u_{\mathrm{t1}}, \\
\text{Transformer (current):} \quad & i_2 = -\tfrac{N_1}{N_2} i_1.
\end{aligned} \tag{3}$$

Then the same short/open circuit procedure as before gives the Thevenin equivalent,

$$U_{\mathrm{T}} = -u_1 \frac{N_2}{N_1} \left( \frac{\mathrm{j}\omega L}{\frac{N_2^2}{N_1^2} R + \mathrm{j}\omega L} \right) = \left( \frac{\mathrm{j}\omega L N_1 N_2}{N_2^2 R + \mathrm{j}\omega L N_1^2} \right) \hat{U}\angle 4\pi/3$$

$$Z_{\mathrm{T}} = \frac{\mathrm{j}\omega L \frac{N_2^2}{N_1^2} R}{\frac{N_2^2}{N_1^2} R + \mathrm{j}\omega L} = \frac{\mathrm{j}\omega L R}{R + \mathrm{j}\omega L \frac{N_1^2}{N_2^2}}$$

The above equations, particularly when expressed in the intermediate form before "simplifying", show how one could have come to the same conclusion by a more step-by-step method. The impedance is a parallel connection of the inductive reactance and the resistance after moving the resistor to the secondary side. The voltage is found by the same principle, using voltage division.

**Exercise 2**

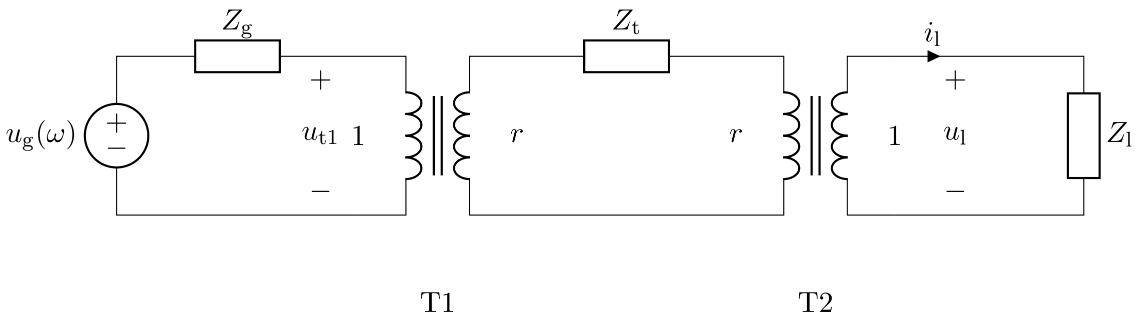
Ideal transformers: practically relevant example.

This question gives practice at:
* Handling transformers in circuit analysis.
* Taking advantage of complex notation and computers, for making the calculations very easy.
* Calculating with Octave/Matlab (surely useful in later courses).
* AC power calculations.
It also introduces some genuine-looking numeric values, and demonstrates the value of high voltage transmission.

The following diagram shows an ac generator, modelled as voltage source $u_{\mathrm{g}}$ with series impedance $Z_{\mathrm{g}}$. It is connected to what we will call the primary side of a transformer T1, which has a turns ratio of 1:$r$ ($r$ on the other side). We'll assume the transformers are ideal.

This 1:$r$ format is a common way of specifying a transformer when one does not care about actual numbers of turns, but just about the ratio: the actual number of turns even in an inideal transformer is anyway not very meaningful unless one also knows details of the magnetic circuit. If T1 instead had $N_1$ defined on the left, and $N_2$ on the right, then $r = N_1/N_2$.

The transformer's secondary winding is connected through a power transmission line with series impedance $Z_{\mathrm{t}}$ to the primary of another transformer T2, with ratio $r$:1. The secondary of T2 is connected to a load, which is an impedance $Z_{\mathrm{l}}$.




This is a good (simple) model of how most of our electric power gets to us: it is generated at one voltage, then "stepped up" to a higher voltage for transmission, then "stepped down" to a low voltage for final use.

We will use the following values of the impedances:
$Z_{\mathrm{g}} = \mathrm{j}1\,\Omega$.
$Z_{\mathrm{t}} = (2 + \mathrm{j}20)\,\Omega$.
$Z_{\mathrm{l}} = 25\,\Omega$.

The line impedance $Z_{\mathrm{t}}$ has been chosen to be representative of a 400 kV overhead line of 100 km length, from `UK-NG` system data. This type of line would normally carry a higher power than we consider here, such as 1 GW (and it would be a three-phase line (another Topic). The load has been chosen as a nice convenient number; you will notice it is entirely resistive, perhaps by including ideal power-factor correction! The generator impedance is reactive, which is a good approximation of a real generator.

In the following, you are welcome to work efficiently to find a numeric answer. You don't need to develop a fully simplified complex expression that gives the sought variable directly from just the given variables. In my opinion, the most efficient way to do this sort of calculation is to use an environment such as Octave/Matlab, and define helpfully named variables for all the given information; then do your calculation in several small steps, defining further intermediate variables, until you reach the solution. This way, the program handles the complex numbers, and you see a symbolic view of what the calculation does, so you can easily change an input value or modify the calculation, or check the result at intermediate stages. It's convenient to put the commands in a script, and run it all at once.

**a)** Let $r = 1$. Assume that a power of 100 MW is delivered to the load $Z_{\mathrm{l}}$.
This is a convenient type of question. You will see that the information allows you to find the currents *and* voltages at the right-hand side of the circuit; then you can successively work towards the left, filling in the currents and voltages in other places to find the requested values. Part 'd' is a more difficult situation.

i) Find the active power that is being lost in the line impedance $Z_{\mathrm{t}}$.

ii) Find the voltage magnitude of the generator, $|u_{\mathrm{g}}|$.
Find also the phase-angle $\angle u_{\mathrm{g}}$, using the load voltage as the reference angle, i.e. define $\angle u_{\mathrm{l}} = 0$; for this you can assume "dots" at the top of all four transformer-windings.

**b)** Now we change just one thing, compared to task 'a': Let $r = 4$.
Calculate the same two things as in task 'a', assuming that the generator voltage has now been adjusted so that there is still a power of 100 MW delivered to the load.

**c)** Comparing the answers to tasks 'a' and 'b', do you see that the transformers and high-voltage transmission serve a useful purpose? Bear in mind that power losses in the line's resistance are proportional to the *square* of the current, and that transforming to $r$ times higher voltage therefore reduces current by $r$ and power loss by $r^2$.

**d)** Let $|u_{\mathrm{g}}| = 50\,\mathrm{kV}$, and $r = 2$. Use the voltage $u_{\mathrm{g}}$ as the phase reference (zero).
What are the voltage, current and complex power at the load: $u_{\mathrm{l}}$, $i_{\mathrm{l}}$ and $S_{\mathrm{l}}$?
Assume, as before, that there are dots at the tops of the windings: this is necessary for calculating phase angles, but not important for the power.

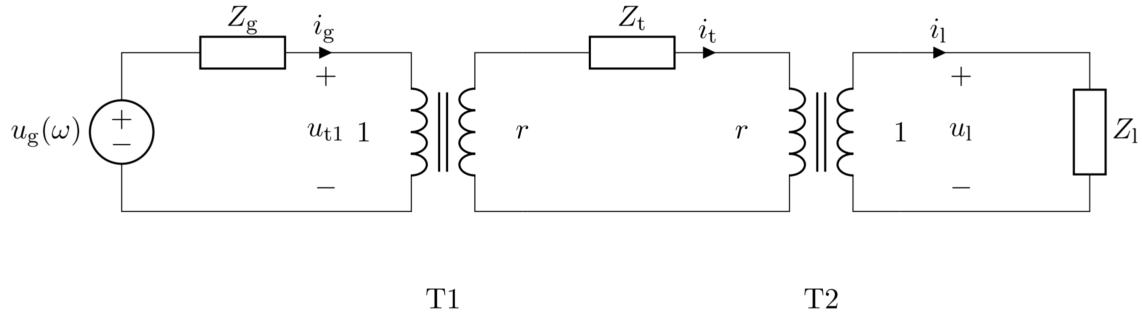
This is harder than the earlier parts, because you can't just work sequentially from one side to the other. You have to consider the right parts of the circuit in order to find the current at the left part … and the left part in order to find what happens in the right part!

It can be done by converting impedances to the other sides of transformers, to find the current from the source, then working from left to right. Alternatively, it can be done by writing and solving equations for each of the three separate electric circuits *and* the relations that the transformers impose between them.

**Answer 2**

Ideal transformers: practically relevant example.

The following diagram shows an ac generator, modelled as voltage source $u_g$ with series impedance $Z_g$. It is connected to what we will call the primary side of a transformer T1, which has a turns ratio of 1:$r$ ($r$ on the other side). The transformer's secondary winding is connected through a power transmission line with series impedance $Z_t$ to the primary of another transformer T2, with ratio $r$:1. The secondary of T2 is connected to a load, which is an impedance $Z_l$.




This is a good (simple) model of how most of our electric power gets to us: it is generated at one voltage, then "stepped up" to a higher voltage for transmission, then "stepped down" to a low voltage for final use.

The impedances are: $Z_g = \mathrm{j}1\,\Omega$, $\;Z_t = (2+\mathrm{j}20)\,\Omega$, $\;Z_l = 25\,\Omega$.

We will assume that there are "dots" at the tops of all the transformer windings. This does not matter for the main questions about power loss, but it does matter for getting phase-angles correct.

**a)** Start with ratio $r = 1$, which means the ideal transformers change nothing in the circuit: the voltages or currents are the same on both sides.

A power of 100 MW is delivered to the load $Z_l$.
The complex power into an impedance $Z_l$ with current $i_l$ flowing in it is $|i_l|^2 Z_l$.
The given value of $Z_l$ is purely resistive (real), so there is no reactive power.
From this, we see that $100\,\mathrm{MW} = |i_l|^2 25\,\Omega$, so $i_l = 2\,\mathrm{kA}$.

**i]** Find the active power lost in the line impedance.
This can be found through the same equation as was used for the load power, by taking the real part of $|i_t|^2 Z_t$. This is $|i_t|^2\,\Re\{Z_t\}$. The transformer equation for T2 tells us that $i_t = i_l/r$. Substituting this, the power loss in the line is

$$P_{t,\mathrm{loss}} = |i_t|^2\ \Re\{Z_t\} = \left|\frac{i_l}{r}\right|^2\ \Re\{Z_t\} = \left(\frac{2\,\mathrm{kA}}{1}\right)^2 2\,\Omega = 8\,\mathrm{MW}.$$

**ii]** Find the voltage magnitude and phase-angle of the generator, $u_g$.
Use the load voltage $u_l$ as the reference angle (zero).

We know the load's impedance and current and therefore also its voltage,

$$u_l = i_l Z_l = 2\,\mathrm{kA}\cdot 25\,\Omega = 50\,\mathrm{kV}.$$

Then the transformer equations for T2 and T1, along with KVL in the middle loop, give the voltage $u_{t1}$ and current $i_g$,

$$u_{t1} = \frac{1}{r}(ru_l + Z_t i_t) = \frac{1}{r}(ru_l + Z_t i_l/r)$$

$$i_g = (i_l/r)r = i_l$$

One more step of KVL leads to the voltage at the source,

$$u_g = u_{t1} + Z_g i_g = \frac{1}{r}(ru_l + Z_t i_l/r) + Z_g i_l.$$

Using the relation $u_l = i_l Z_l$, the expression for $u_g$ can be expressed in terms of $u_l$,

$$u_g = \frac{1}{r}\left(ru_l + Z_t\frac{u_l}{rZ_l}\right) + Z_g\frac{u_l}{Z_l} = u_l\left(1 + \frac{Z_t}{r^2 Z_l} + \frac{Z_g}{Z_l}\right).$$

The load voltage is known to be $u_l = 50.0\underline{/0}$ kV.
Inserting the numeric value, the above expression gives $u_g = 68.4\underline{/37.9^\circ}$ kV.

In Octave/Matlab, this is calculated as:

```
Zg = 1j,   Zt = 2 + 20j,   Zl = 25,   Sl = 100e6,   r = 1
ul = sqrt( conj(Zl) * Sl )

% ONE WAY:  using the formula derived above
ug = ul * ( 1 + Zt/Zl/r^2 + Zg/Zl )
abs(ug), angle(ug)*180/pi
%>  68.4 kV,   at +37.9 degrees.

% OTHER WAY:  working step by step
ul = sqrt( conj(Zl) * Sl )
%      50000
il = ul/Zl
%        2000
u_T2pri = r*ul    % primary (left) of T2
%      50000
it = il/r
%       2000
u_T1sec = u_T2pri + it*Zt
%     5.4000e+04 + 4.0000e+04i
ig = it*r
%       2000
ut1 = u_T1sec / r
%     5.4000e+04 + 4.0000e+04i
ug = ut1 + ig*Zg
%     5.4000e+04 + 4.2000e+04i
abs(ug), angle(ug)*180/pi
%>  68.4 kV,   at +37.9 degrees
```

**b)** Now the ratios are $r = 4$. There is still 100 MW delivered to the load.

**i]** Using the same reasoning as in part 'a',

$$P_{\mathrm{t,loss}} = \left|\frac{i_\mathrm{l}}{r}\right|^2 \Re\{Z_\mathrm{t}\} = \left(\frac{2\,\mathrm{kA}}{4}\right)^2 2\,\Omega = 0.5\,\mathrm{MW}.$$

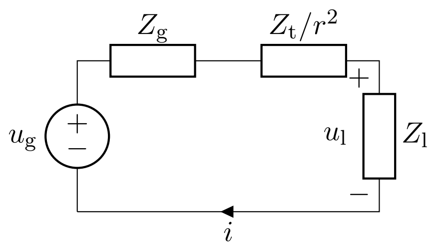
**ii]** Using the same program as in 'a', but with $r = 4$, the result is that $u_\mathrm{g} = 50.45\underline{/5.1°}$ kV.

```
Zg = 1j,    Zt = 2 + 20j,    Zl = 25,    Sl = 100e6,   r = 4
ul = sqrt( conj(Zl) * Sl )
ug = ul * ( 1 + Zt/Zl/r^2 + Zg/Zl )
abs(ug), angle(ug)*180/pi
%>  50.45 kV,  at +5.1 degrees.
```

**c)** Comparison of answers (mainly losses) to tasks 'a' and 'b'.

Simply increasing the voltage by 4× reduced the power losses in the long line by 16×. When 400 kV is used for bulk transmission of electricity, instead of the 400 V that is used for normal consumers, the overall transformation ratio is $r = 1000$. The effect on losses is therefore *very* big: $r^2 = 10^6$. High voltages clearly have a great advantage for long-distance power transmission. On the other hand, high voltages demand more expensive equipment, and result in more power-loss around the wires in the air (the "crackle" sound, particularly when wet). This is why it is worth paying for several transformers between different voltage levels in the system, so that each level has a suitable voltage for the power it handles and its geographic extent; the choice of voltage is a compromise between various factors, including losses in the line-resistance, cost and size of cables, switches, etc.

The voltage magnitude and angle at the source were also interesting to compare. With a higher voltage in the transmission part of the system, the excess voltage (above 50 kV) and the angle became smaller. We can see this by moving the line impedance $Z_\mathrm{t}$ to the other side of one of the transformers: it would be scaled by the factor $1/r^2$. This expression shows that a higher choice of $r$ rapidly reduces the effective line-impedance seen from the lower-voltage sides of the transformers. Note that in our simple case with no parallel components, and with the two transformers having inverse ratios, we could have replaced the whole circuit with just the following:




**d)** Let $|u_\mathrm{g}| = 50\,\mathrm{kV}$, and $r = 2$. Use the voltage $u_\mathrm{g}$ as the phase reference (zero).
What are the voltage, current and complex power at the load: $u_\mathrm{l}$, $i_\mathrm{l}$ and $S_\mathrm{l}$?

One approach is to do the same as in the equation-based solution of Q3: write and solve the equations imposed by KVL, KCL, Ohm's law and the transformer equations.

However, a simplified equivalent has already been suggested (see the diagram above, in part 'c'), where the transformers have been eliminated; let us try this first, as it is easy to solve.

We have $u_\mathrm{g} = 50.0\underline{/0}$ kV, and $r = 2$. The current around the loop is

$$i = \frac{u_\mathrm{g}}{Z_\mathrm{g} + Z_\mathrm{t}/r^2 + Z_\mathrm{l}} = \frac{50\,\mathrm{kV}}{1\mathrm{j}\,\Omega + \frac{(2+20\mathrm{j})\Omega}{2^2} + 25\,\Omega} = 1.909\underline{/-13.2°}\text{ kA}.$$

The voltage across the load is

$$u_\mathrm{l} = iZ_\mathrm{l} = 25\,\Omega \times 1.909\underline{/-13.2°}\text{ kA} = 47.7\underline{/-13.2°}\text{ kV}$$

The complex power in the load is $|i|^2 Z_\mathrm{l}$, which is

$$S_\mathrm{l} = \left|\frac{u_\mathrm{g}}{Z_\mathrm{g} + Z_\mathrm{t}/r^2 + Z_\mathrm{l}}\right|^2 Z_\mathrm{l} = \left(\frac{50\,\mathrm{kV}}{\left|1\mathrm{j}\,\Omega + \frac{(2+20\mathrm{j})\Omega}{2^2} + 25\,\Omega\right|}\right)^2 \times 25\,\Omega \; = \; 91.1\,\mathrm{MW}.$$

There are various other ways of doing the calculation, without using the idea of simplifying away the transformers, or of writing all the equations. One is to define a voltage or current at the load, then express the source voltage from this; the equation can then be rearranged to find the variable at the load from a known voltage at the source. In our case, the solution has already been done in part 'a', to find `ug` from `ul`. Rearranging the equation to get `ul`,

```
r = 2, ug = 50e3,  Zg = 1j,  Zt = 2+20j,  Zl = 25
ul = ug / ( 1 + Zt/Zl/r^2 + Zg/Zl )   %  47.7 kV at -13.2 degrees (w.r.t. ug)
il = ul / Zl          %    1.909 kA at -13.2 degrees
Sl = ul * conj(il)    %    91.1 + j0  MVA
```