KTH, Electric Circuit Analysis, EI1120

Topic 13 Practice Exercises

CALCULATIONS IN BALANCED THREE-PHASE CIRCUITS

The three-phase topic adds a few further definitions, like what "line-voltage" means. Other than this, the three-phase analysis only *needs* the phasor analysis and ac power calculations that were studied in previous topics, with occasional appearance of transformers.

However, symbolic solutions of three-phase systems "from first principles" would be a *lot* of work: there are big advantages in *solution simplicity* in three-phase circuits if one recognises symmetries and power-based shortcuts, and knows some useful rules, which often involve $\sqrt{3}$!

Practice is needed, in order to get competence and confidence at using these shortcut methods, and to become very familiar with how the voltages and currents in Δ and Y connections are related. The three-phase exercises also provide a chance to practice more with ac phasor analysis, ac power and transformers: that's helpful even for solving non-three-phase problems.

The following exercises work through a range of features of **balanced** three-phase circuits: voltage and current, power calculation, numeric or symbolic, magnitude-only or complete phasor. There is a deliberate mix of notation between questions (e.g. ω or f, U denoting phase-voltage or line-voltage, calling Y-connection 'star' or 'wye'), in order to familiarize you with different conventions.

Some of the exercises are purely abstract, but others are hoped to feel relevant to practical situations you might have observed. They range from easy up to harder than any exam question we've made (the last few Exercises).

Some past exam questions are listed below. A common choice is a balanced situation based on power and compensation, and an unbalanced situation with some voltage or current magnitude to be found. Something of this sort is expected again.

2014-03˙E˙tenta Q6 balanced line and load: find voltage at load

2012-03˙E-EM˙tenta Q4 balanced; find current phasors

2014-05˙EM˙omtenta Q8 Y-D power-factor correction, and current phasor

2014-03˙EM˙tenta Q8 Y-D power-factor correction, and current phasor

2015-03˙EM˙tenta Q9 more Y-D and power-factors

2016-06˙EM˙omtenta Q9-a more Y-D and power-factors

2016-03˙EM˙tenta Q9-ab more Y-D and power-factors

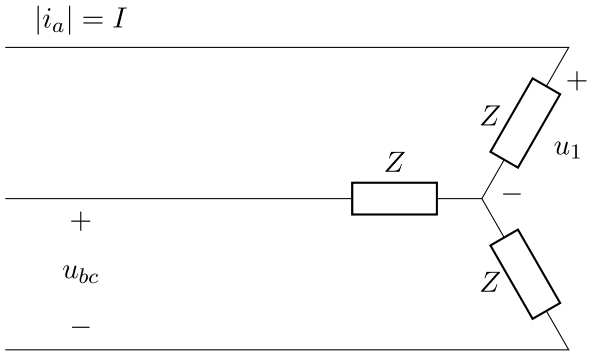
2015-06˙EM˙omtenta Q9-ab more Y-D and power-factors

2015-06˙E˙omtenta2 Q9 three-phase multichoice

2015-03˙E˙tenta2 Q9 three-phase multichoice

Exercise 1

The terminals at the left in this diagram are connected to a three-phase voltage source.



Conventions: unless told otherwise, in these exercises you can assume that:

- A "source" is a voltage source.
- Components are ideal: e.g. a voltage source has no impedance.
- A three-phase source is *balanced*: the sources of the different phases have the same magnitudes, and are phase-shifted 120° from each other.
- Impedances used e.g. as loads or lines are also balanced: all equal.

We see that the magnitude of line-current is defined as I.
The impedances in the Y-connected load are all Z.
These are the two 'known quantities'.

a) Find the *magnitude* of the load's phase-voltage $|u_1|$.

b) And find the magnitude of the line-voltage $|u_{bc}|$.

Yes, the same results for magnitude should be found for *any* of the three phases of the load (part a) or between any pair of the three conductors of the line (part b). We just happen to have chosen a particular phase of the load and a particular pair of lines, on which to mark the voltages.

Answer 1

a) This load is Y-connected. That means that each phase of the load connects between one of the lines and a common node (neutral). The line current and phase current are therefore the same. This can be seen by KCL, looking at the diagram. Or it can be remembered as a feature of a Y-connection.

Thus, the magnitude of the load's phase-voltage is easily found from the magnitudes of phase-current and impedance[1] by Ohm's law:

$$|u_1| = I|Z|.$$

We have no further information about Z, so we can't express this any more clearly than by writing the magnitude symbol, $|Z|$.

b) The load where u_1 is defined is Y-connected to the line, so the familiar number $\sqrt{3}$ will relate the phase-voltage magnitude to the line-voltage magnitude $|u_{bc}|$:

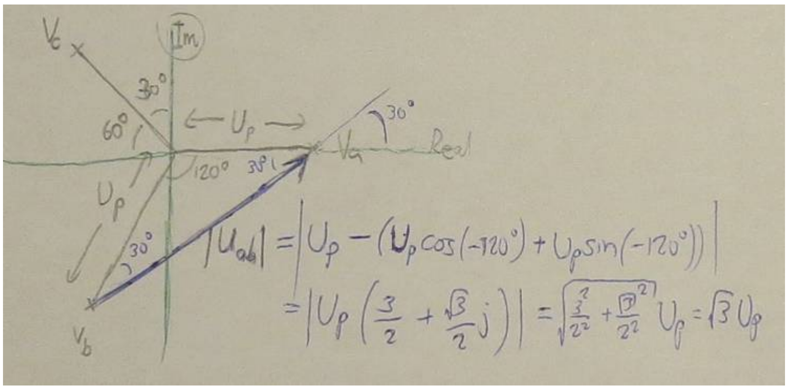
$$|u_{bc}| = \sqrt{3}\,|u_1| = \sqrt{3}\,I\,|Z|.$$

Unless asked to 'prove' this, it's reasonable just to state the above solution by using the well-known ratio $\sqrt{3}$ between the line and phase voltages in a star-connection (Y-connection).

If you *do* want to prove it from more basic ac principles, then start by defining voltages across the three impedances (e.g. u_1, u_2, u_3, with respect to the middle-point of the load), and noting that these voltages have phase-shifts of 120° due to connection of a balanced load to a balanced 3-phase source.

We can define u_P as the phase-voltage magnitude ($u_\mathrm{P} = |u_1|$), then any line voltage can be expressed as $u_\mathrm{P}\underline{/\phi} - u_\mathrm{P}\underline{/\phi{-}120°}$, or equivalently as $u_\mathrm{P}\mathrm{e}^{\mathrm{j}\phi} - u_\mathrm{P}\mathrm{e}^{\mathrm{j}(\phi-2\pi/3)}$, where ϕ is the angle of one of the phase voltages with respect to whatever we've chosen as the reference angle (we aren't given any specific phase angles in the question).

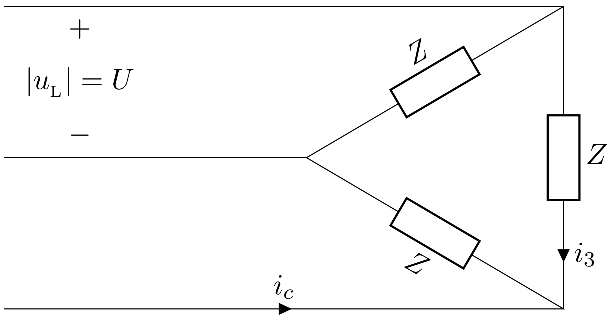
This magnitude, by application of a little trigonometry, is $\sqrt{3}u_\mathrm{P}$.



[1] You might be worried by the claim that one can ignore the phase angles when multiplying magnitudes to find a magnitude? If we had an expression like $u_x + u_y$ then the magnitude of the result *does* depend on the phase-angles of these quantities: $|u_x + u_y|$ *cannot* be assumed to be $|u_x| + |u_y|$. But when we multiply or divide complex numbers, the magnitudes and angles *can* be treated separately: $A\underline{/\alpha}/B\underline{/\beta} = (A/B)\underline{/\alpha - \beta}$.

Exercise 2

At the left (not shown), a three-phase source provides a line-voltage with magnitude U.



Find:

a) The marked phase-current magnitude in the load $|i_3|$.

b) The line-current magnitude, $|i_c|$.

Answer 2

(This is a sort of dual case to the Y-connection question.)

a) For this Δ-connection, the voltage across each phase of the load (each impedance Z) is the same as the line voltage. This can be seen by KVL.

Thus, $|i_3|$ is found from the magnitude of the voltage and of the impedance:

$$|i_3| = \frac{U}{|Z|}.$$

We can't express the magnitude of the impedance[2] any better than $|Z|$, since we have no further information about it.

b) The load where i_3 is defined is Δ-connected, so the familiar number $\sqrt{3}$ will relate this phase-current's magnitude to the line-current magnitude $|i_c|$:

$$|i_c| = \sqrt{3}\,|i_3| = \frac{\sqrt{3}U}{|Z|}.$$

As in the previous question, if you want to 'prove' that this factor of $\sqrt{3}$ arises, you can do so by noting that each line-current is a difference between two phase-currents (if we define all phase currents going the same way around the Δ), and that the phase-currents are phase-shifted from each other by $\frac{2\pi}{3}$.

The factor $\sqrt{3}$ then appears by the same reasoning as in the previous question, because

$$\sqrt{3} = \left|\, 1\underline{/\phi + 0} \;-\; 1\underline{/\phi - (-120°)}\,\right|.$$

This is true for any value of ϕ, and for $+120$ degree or -120 degree shifts between the two phasors.

Further notes:

Due to the assumption of balanced three-phase conditions, the above magnitudes would be valid for any of the line currents (i_a, i_b, i_c) and any of the phase currents (i_1, i_2, i_3).

[2] We often see impedances in rectangular form, for example $Z = R + \mathrm{j}\omega L$. Sometimes it's instead desirable to express them as magnitude and phase, i.e. in polar form, as we also sometimes do with phasors. Despite the similarity of how we represent phasors and impedances, there is a fundamental difference between them: a phasor represents the magnitude and phase of a steady-state sinusoidally time-varying quantity (like voltage or current), whereas an impedance expresses the relation (magnitude ratio and phase-shift) *between* a voltage and current.

Exercise 3

A three-phase generator supplies a voltage of 440 V and its maximum rated current is 657 A.

a) What is its apparent power output when it is running at its maximum current?

b) How much active power (sometimes called real power or true power) would it supply to a load that has PF $= 0.8$, if loaded to its maximum current?

Notes:
You can assume the stated voltage is a line-voltage; in fact, 440 V is a common line-voltage, giving a line-neutral voltage (phase-voltage[3]) of around 250 V.
The voltage and current are sure to be rms values; that's what is always used in power applications.

In case you're interested: the above details are from a specification of generators [pdf-link] of the sort that are hired for temporary power, building sites, etc. There are some quite large ones, e.g. with 3.3 kA currents. The document helpfully explains how to calculate three-phase power: notice their factor 1.732, which we call $\sqrt{3}$. The word 'alternator' is often used for ac generators.

[3] Line-neutral voltage is often called phase-voltage when we're considering networks, cables, connections, etc. But at this stage in the exercises, we're still trying to keep a textbook-style terminology where 'phase' voltages or currents are strictly the voltages and currents of the single-phase sources or impedances that make up the three-phase sources or impedances.

Answer 3

a) The standard formula for balanced three-phase apparent power is[4]

$$|S| = \sqrt{3}IU,$$

where I and U are magnitudes of line current and line voltage. Hence

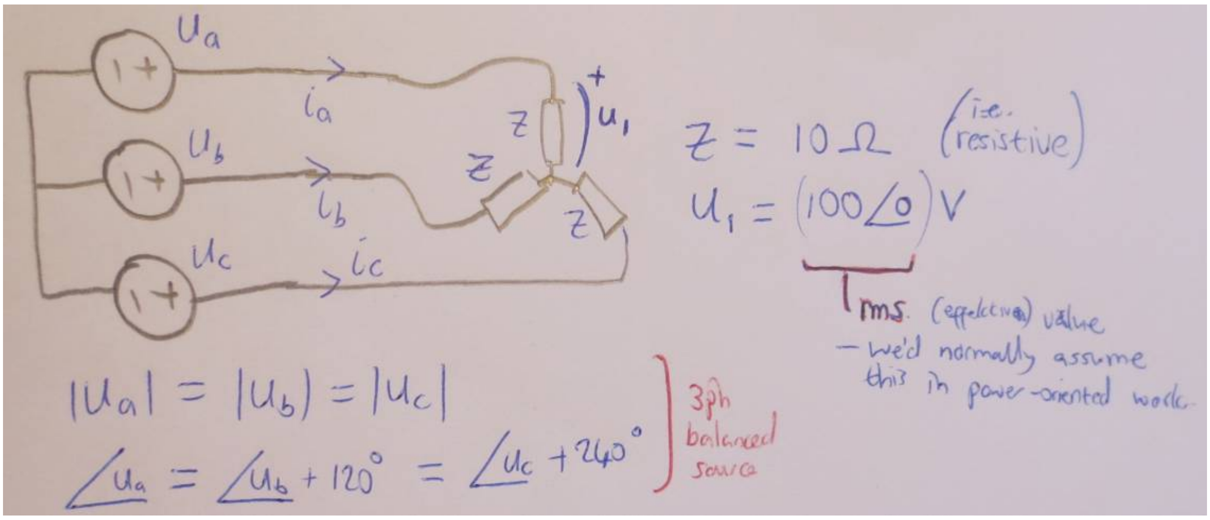
$$|S| = \sqrt{3} \cdot 440\,\mathrm{V} \cdot 657\,\mathrm{A} = 500\,\mathrm{kVA}.$$

b) The power factor is the ratio $\textsc{pf}= \frac{P}{|S|}$. The maximum active power P available from this generator (within its rated maximum current) is then $0.8|S|$:

$$P = 0.8 \cdot 500\,\mathrm{kVA} = 400\,\mathrm{kW}.$$

[4] This is equivalent to $3\frac{1}{\sqrt{3}}UI$, which helps explain where the equation comes from. For example, if you have a 400 V system, then each phase of a Y-connected load will get 230 V, and will have the same current as the line current; each phase in the load then has apparent power $IU/\sqrt{3}$ where I and U are line quantities, so we write the total power for the three phases as $\sqrt{3}UI$.

Exercise 4



Given the stated values of the marked voltage u_1 and the phase-impedances Z of the load, find:

a) The voltage magnitude of phase ‘a’ of the source, $|u_a|$.

b) The current magnitude in line ‘c’, $|i_c|$.

c) The total active power consumed by the three-phase load.

d) The voltage u_c as a phasor (angle as well as magnitude).

e) The current i_b as a phasor.

Answer 4

Notice that these are just a balanced Y-connected source and Y-connected load, with no line-impedance between. Each phase of the source therefore has the same voltage and current as the corresponding phase of the load.

a) By 3-phase symmetry of the source and load, we expect the middle points ('star-points' or 'neutral points') of the source (node at left) and of the load (node in the centre of the load) to have the same potential.

Thus, by KVL, $u_1 = u_a$, so $|u_a| = 100\,\mathrm{V}$.

If you want to prove it better, try calling the left node zero potential, and writing a nodal equation (KCL in the middle of the load) to find the potential v_n at the middle of the node: $\frac{v_\mathrm{n}-u_a}{Z} + \frac{v_\mathrm{n}-u_b}{Z} + \frac{v_\mathrm{n}-u_c}{Z} = 0$. With the voltages $u_{a,b,c}$ being a balanced three-phase set, so that $u_a + u_b + u_c = 0$, we see that $v_\mathrm{n} = 0$, the same potential as the left node.

b) The current in line 'c', i_c, is the same as in the load impedance at the bottom right.
By similar reasoning to subquestion 'a', we know the voltage across that impedance is u_c, and so the current is $i_c = u_c/Z$. We don't need to include the angles, because we only were asked about the magnitude.

Using the given numbers, $|i_c| = \frac{|u_c|}{Z} = \frac{100\,\mathrm{V}}{10\,\Omega} = 10\,\mathrm{A}$.

c) The complex power into a single impedance Z subjected to voltage magnitude (rms, as usual) u, is $S = u^2/Z^*$. We have 3 identical impedances, each purely real, subjected to voltage magnitude $100\,\mathrm{V}$. The solution is therefore purely real, being only active power.

Hence, the load's power is $P = 3\frac{(100\,\mathrm{V})^2}{10\,\Omega} = 3\,\mathrm{kW}$.

Alternatively, work with the current found in subquestion 'b'.
The source is Y-connected, so the line voltage magnitude is $\sqrt{3} \cdot 100\,\mathrm{V} = 173\,\mathrm{V}$.
The line current magnitude is $10\,\mathrm{A}$.
The apparent power (skenbar effekt) to a balanced three phase load is $|S| = \sqrt{3}ui$, where u and i are the magnitudes of line voltage and current.
Thus, $|S| = \sqrt{3}\,(\sqrt{3} \cdot 100\,\mathrm{V})\,10\,\mathrm{A} = 3\,\mathrm{kVA}$.
Knowing that the load is purely resistive, we can state $P = |S| = 3\,\mathrm{kW}$.

d) Only one phase-angle is defined in the question: $\angle u_1 = 0$.
This let us know that $\angle u_a = 0$ (subquestion 'a').
The diagram states that $\angle u_a = \angle u_c + 240°$, which tells us that $u_c = (100\angle{-240°})\,\mathrm{V}$.

e) By similar reasoning to subquestion 'd', we see $u_b = (100\angle{-120°})$ V.
The system is balanced, so it has the same potential at the neutral points of the source and load (even though there is no neutral conductor). The source u_b the resistor it connects to will therefore have the same voltage, so by Ohm's law the current i_b is

$$i_b = \frac{u_b}{10\,\Omega} = \frac{(100\angle{-120°})\ \mathrm{V}}{10\,\Omega} = (10\angle{-120°})\ \mathrm{A}.$$

Exercise 5

A 400 kV overhead transmission line carries 1200 MW from a large power station.

What is the current in this line if the power is transferred at unity power-factor ($\text{PF} = 1$)?

Yes, we assume it's a three-phase line, and that it's running at its rated voltage, and that that's a line-voltage magnitude, and that it's in rms scale, ...

Answer 5

Unity PF implies $P = |S|$.

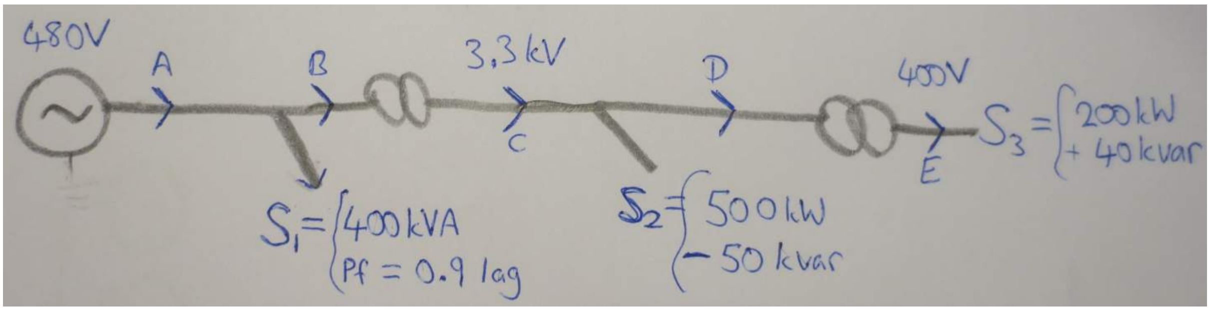
So,

$$I = \frac{P}{\sqrt{3}U} = \frac{1.2{\times}10^9\,\mathrm{W}}{\sqrt{3}\cdot 4{\times}10^5\,\mathrm{V}} = 1732\,\mathrm{A}.$$

Exercise 6

This is a “single-line diagram” (SLD) of a 3-phase system.

Each drawn line (e.g. the horizontal line where ‘A’ is marked) represents a set of three (or four, if there’s a neutral) conductors that form a three-phase circuit — for example, a cable or overhead line. We assume it’s all balanced: balanced source, balanced lines, balanced loads.



There are two transformers, shown by symbols made from two linked circles.

Three loads are connected, with complex powers S_1, S_2 and S_3 defined in the diagram.

The source is shown at the left (this is a common way to show a three-phase or single-phase voltage source in a SLD).

The lines can be treated as having no impedance, and the transformers as being ideal.

Four points are marked on the lines: A, B, C and D.

For each of these points, calculate the complex power flowing in the direction shown by the arrow, and the line-current magnitude.

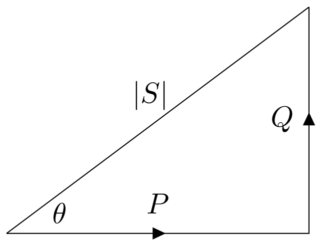
Answer 6

To make it easy to sum the different loads, it is convenient to convert all the loads to rectangular form, i.e. complex power expressed as $S = P + \mathrm{j}Q$.

The definitions of apparent power, power factor and power factor angle are that:

$$|S|^2 = P^2 + Q^2$$

$$\text{PF} = \frac{P}{|S|} = \cos\theta$$



from which various other popular relations can be derived, such as

$$\frac{Q}{P} = \tan\theta, \quad Q = P\sqrt{\frac{1}{(\text{PF})^2} - 1}, \quad Q = P\,\tan\left(\cos^{-1}(\text{PF})\right), \quad Q = S\sqrt{1 - (\text{PF})^2}$$

From such relations, we find the complex powers as:

$$S_1 \quad = \quad 400\,\text{kVA}\ (\text{PF}\ 0.9\ \text{lag}) = 360.0\,\text{kW} + \mathrm{j}174.4\,\text{kvar}$$
$$S_2 \quad = \quad 500\,\text{kW} - 50\,\text{kvar}$$
$$S_3 \quad 200\,\text{kW} + \mathrm{j}40\,\text{kvar}$$

Notice the convenience that, assuming ideal transformers, the powers on both sides of the transformer are the same: what goes in comes out. So, for power flows we don't need to consider the transformer's details, such as its ratio or its Y or Δ connections.

All that is needed is to find the power flow at each marked point, from the sum of the 'downstream' (to the right) powers. Then the line current magnitude can be found by the usual relation of apparent power to line voltage and current magnitude, $|S| = \sqrt{3}ui$,

A) $S_\text{A} = S_1 + S_2 + S_3 = (1060{+}\mathrm{j}164.4)\,\text{kVA}, \quad |I_\text{A}| = \frac{|S_\text{A}|}{\sqrt{3}\cdot 480\,\text{V}} = 1290\,\text{A}$

B) $S_\text{B} = S_2 + S_3 = (700{-}\mathrm{j}10)\,\text{kVA}, \quad |I_\text{B}| = \frac{|S_\text{B}|}{\sqrt{3}\cdot 480\,\text{V}} = 842.1\,\text{A}$

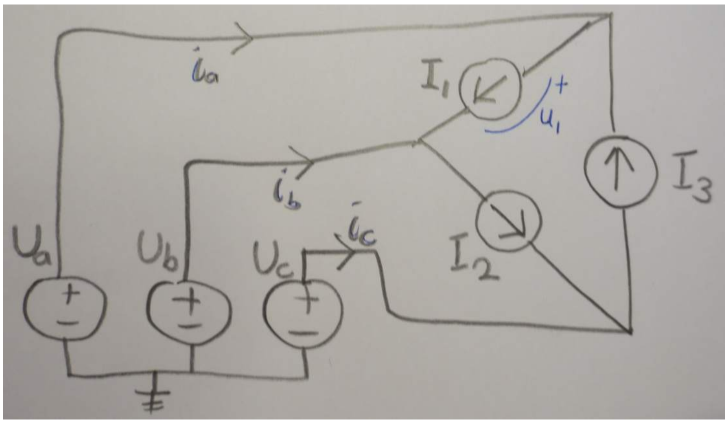
C) $S_\text{C} = S_2 + S_3 = (700{-}\mathrm{j}10)\,\text{kVA}, \quad |I_\text{C}| = \frac{|S_\text{C}|}{\sqrt{3}\cdot 3.3\,\text{kV}} = 122.5\,\text{A}$

D) $S_\text{D} = S_3 = (200{+}\mathrm{j}40)\,\text{kVA}, \quad |I_\text{D}| = \frac{|S_\text{D}|}{\sqrt{3}\cdot 3.3\,\text{kV}} = 35.7\,\text{A}$

E) $S_\text{E} = S_3 = (200{+}\mathrm{j}40)\,\text{kVA}, \quad |I_\text{E}| = \frac{|S_\text{E}|}{\sqrt{3}\cdot 400\,\text{V}} = 294.4\,\text{A}$

Exercise 7

On the left is a three-phase voltage source. On the right is a three-phase current source: we don't often consider current sources in three-phase circuits, as most power-related sources are good approximations of voltage sources; but current sources do have some relevance, particularly in situations with power-electronic converters.



Let $U_a = U\angle 0$, and $I_1 = I\angle 0$. The other sources are phase-delayed (lagging) by radian angles of $2\pi/3$ (sources b and 2) and $4\pi/3$ (sources c and 3).
The known quantities to use in the solutions are U and I, not U_a, I_1, etc. You can assume these are rms values.

a) Find the phasor i_a.

b) Find the phasor u_1.

c) What complex power S is delivered (sent out) by the 3-phase voltage source?

d) Express this (complex-)power S as apparent power and power-factor.

e) What complex power is delivered by the source I_2?

Notation conventions: Some people show complex power (a complex value) as **S**, and apparent power (*skenbar effekt*, a real value) as S. Here, we're using S for complex power, and including an explicit 'magnitude' symbol, $|S|$, for apparent power.

Answer 7

a) Find the phasor i_a.
By KCL in the top node, $i_a = I_1 - I_3$. Using the given details about phase-angles and magnitudes,

$$i_a = I_1 - I_3 = I\underline{/0} - I\underline{/-240°} = I\underline{/0} + I\underline{/-60°} = \sqrt{3}I\underline{/-30°}.$$

You can see this by sketching a diagram. If you prefer rectangular (kartesisk) thinking, consider that

$$i_a = I_1 - I_3 = I \cdot (1 - \cos(-240°) - \mathrm{j}\sin(-240°)) = I\left(\frac{3}{2} - \mathrm{j}\frac{\sqrt{3}}{2}\right),$$

from which

$$|i_a| = I\sqrt{\left(\frac{3}{2}\right)^2 ... + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{3}I \qquad \text{and} \qquad \underline{/i_a} = \tan^{-1}\frac{-1}{\sqrt{3}} = -30°.$$

b) Find the phasor u_1.
By KVL, $u_1 = U_a - U_b$.
A similar reasoning to the above can be used, but remembering that phase b is shifted by $-120°$ from phase a (not $-240°$ as in the above),

$$u_1 = U_a - U_b = U \cdot (1 - \cos(-120°) - \mathrm{j}\sin(-120°)) = U\left(\frac{3}{2} + \mathrm{j}\frac{\sqrt{3}}{2}\right) = \sqrt{3}U\underline{/30°}.$$

c) What complex power S is delivered (sent out) by the 3-phase voltage source?
For the a-phase of the voltage source (U_a), the complex power output S_a is

$$S_a = U_a i_a^* = UI\left(\frac{3}{2} - \mathrm{j}\frac{\sqrt{3}}{2}\right)^* = UI\left(\frac{3}{2} + \mathrm{j}\frac{\sqrt{3}}{2}\right) = \frac{3UI}{2} + \mathrm{j}\frac{\sqrt{3}UI}{2}.$$

By symmetry[5] we expect the same complex power from the other two phases of the three-phase source. The phase-angles of voltage and current are different for these sources (120° shifts) but the voltage and current are *both* shifted the same angle, so the relation ui^* is unchanged.[6] The total complex power is then just three times that of one phase of the source,

$$S = 3S_a = 3\,\frac{UI}{2}\left(3 + \mathrm{j}\sqrt{3}\right) = \frac{9UI}{2}\left(1 + \mathrm{j}\frac{1}{\sqrt{3}}\right).$$

d) Express this power S as apparent power and power-factor.
Note that we can define active and reactive power as $S = P + \mathrm{j}Q$. Then,

$$|S| = \sqrt{P^2 + Q^2} = \frac{9UI}{2}\sqrt{3^2 + \sqrt{3}^2} = 9\sqrt{3}UI,$$

[5]Notice that in a classic "source and impedance" system, symmetry just means balanced source and balanced impedance loads. But in *this* system, where two sources are connected, symmetry requires that both sources were connected with the same "phase rotation". If, for example, any two single-phase voltage sources or any two current sources were swapped, then both the three-phase sources would still be balanced, but they would be connected to each other with opposite phase-rotation: the resulting powers would be different in the different phases, as the complete system would not be symmetric.

[6]No change in complex power when phase-shifting u and i equally: $S = ui^*$, and let $u = |u|\underline{/\alpha}$ and $i = |i|\underline{/-\beta}$. Then if both angles change by δ, $S = |u|\underline{/\alpha + \delta} \cdot |i|\underline{/-\beta - \delta}$, simplifying to $S = |u||i|\underline{/\alpha - \beta + \delta - \delta} = |u||i|\underline{/\alpha - \beta}$.

and

$$\mathrm{PF} = \frac{P}{|S|} = \frac{3}{\sqrt{3^2 + \sqrt{3}^2}} = \frac{\sqrt{3}}{2} = \cos(30^\circ) \text{ lagging.}$$

It is seen to be a lagging power factor because the phase current from the voltage source's a-phase (see subquestion 'a') has a phase of $-2\pi/6$ relative to that source's a-phase voltage. Alternatively, notice that the reactive power output is positive, as if feeding an inductive load. The choice of phase-angles of the two sources is responsible for this.

e) What power is delivered by the source I_2? (Let's call this S_2.)

This source is one of the phases of a three-phase source. By symmetry, each of the phases of a balanced source in a balanced system has the same complex power: $\frac{1}{3}$ of the source's total.

By conservation of complex power (total complex power generated equals total consumed, in a circuit) the three-phase current source in our circuit receives all the power that the three-phase voltage source delivers. Or, we could instead say that the current source delivers all the power that the voltage source receives: it means the same thing, as we don't know the actual direction of complex power flow until we've thought about the phase-angles.

So, using the result from subquestion 'd',

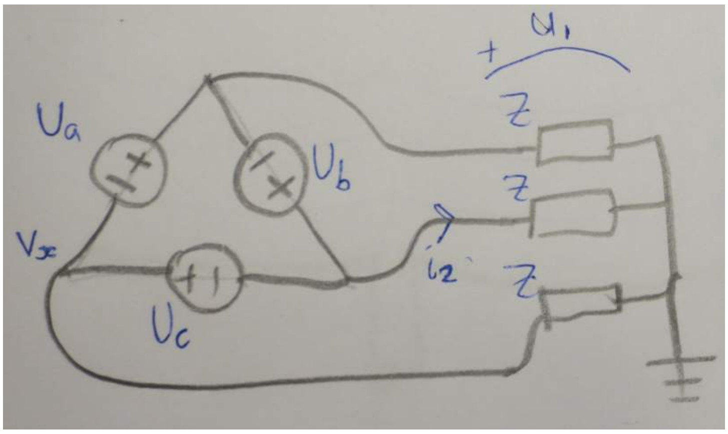
$$S_2 = -\frac{1}{3}S = -\frac{1}{3}\frac{9UI}{2}\left(3 + \mathrm{j}\sqrt{3}\right) = \frac{9UI}{2}\left(-1 - \mathrm{j}\frac{1}{\sqrt{3}}\right),$$

which means that this source is in fact *absorbing* active and reactive power, not delivering it. Seen from the voltage source, it "looks like" a resistive-inductive load. (But if the voltage source phase or magnitude were to change, the load could look different, e.g. like a capacitor, or a power source, or a pure resistor, depending on the relative phase of the sources).

Many choices could be made for the way of expressing the above solutions. As long as any obvious big simplifications have been made, it doesn't matter which form you choose.

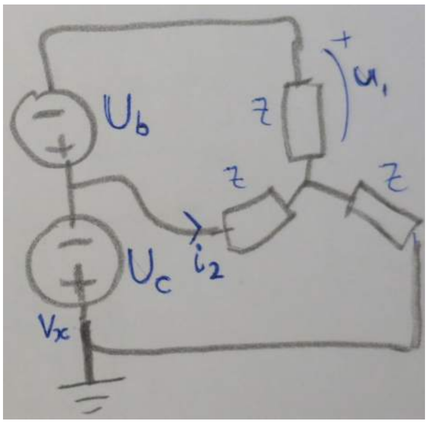
Exercise 8

The known quantities here are phasors U_a, U_b and U_c (forming a three-phase source), and impedance Z. Let the magnitude of each of the phasors be U, i.e. $U = |U_a| = |U_b| = |U_c|$.



a) Find the *magnitudes* of u_1, i_2 and v_x.

b) Repeat the above question, but with the following circuit.



Answer 8

a) Magnitudes of the marked quantities in the upper circuit:

Phase-voltage magnitude in load: $|u_1| = U/\sqrt{3}$.

The source is Δ-connected, so the line voltage (magnitude) equals the voltage (magnitude) of each phase within the source. The load is Y-connected to the line, so its phase voltage is $\frac{1}{\sqrt{3}}$ of the line voltage.

Line current magnitude: $|i_2| = \frac{U}{\sqrt{3}Z}$.

The current magnitude in each phase of the load is $|u_1|/Z$. Due to the Y-connection of the load, the line-current and phase-current have the same magnitude.

The magnitude of the marked potential: $|v_x| = U/\sqrt{3}$.

Since the zero potential is defined as the middle-point ('star-point' or 'neutral-point') of the Y-connected load, the marked potential is just the phase voltage of the lowest phase of the load! This can be seen by 'potentialvandring'.

... continued on next page ...

b) Magnitudes of the marked quantities in the lower circuit:

Phase-voltage magnitude in load: $|u_1| = U/\sqrt{3}$.

Line current magnitude: $|i_2| = \frac{U}{\sqrt{3}Z}$.

The magnitude of the marked potential: $|v_x| = 0$.

The reasoning for these answers is the following.

The two loads are the same, in the first and second circuit. And the two sources are *also* the same, when seen from their three terminals.[7]

Thus, with identically behaving load, source and connection, we would expect the circuits to have the same solutions. The only worry is that the earth node is defined differently between these circuits. But each circuit had only *one* node marked as earth, so the actual voltages, currents and powers are not affected by the earth node: the earth node only determines how we define *potentials*.[8]

The new circuit has the earth point on the node where v_x is defined – so, "by definition", $v_x = 0$.

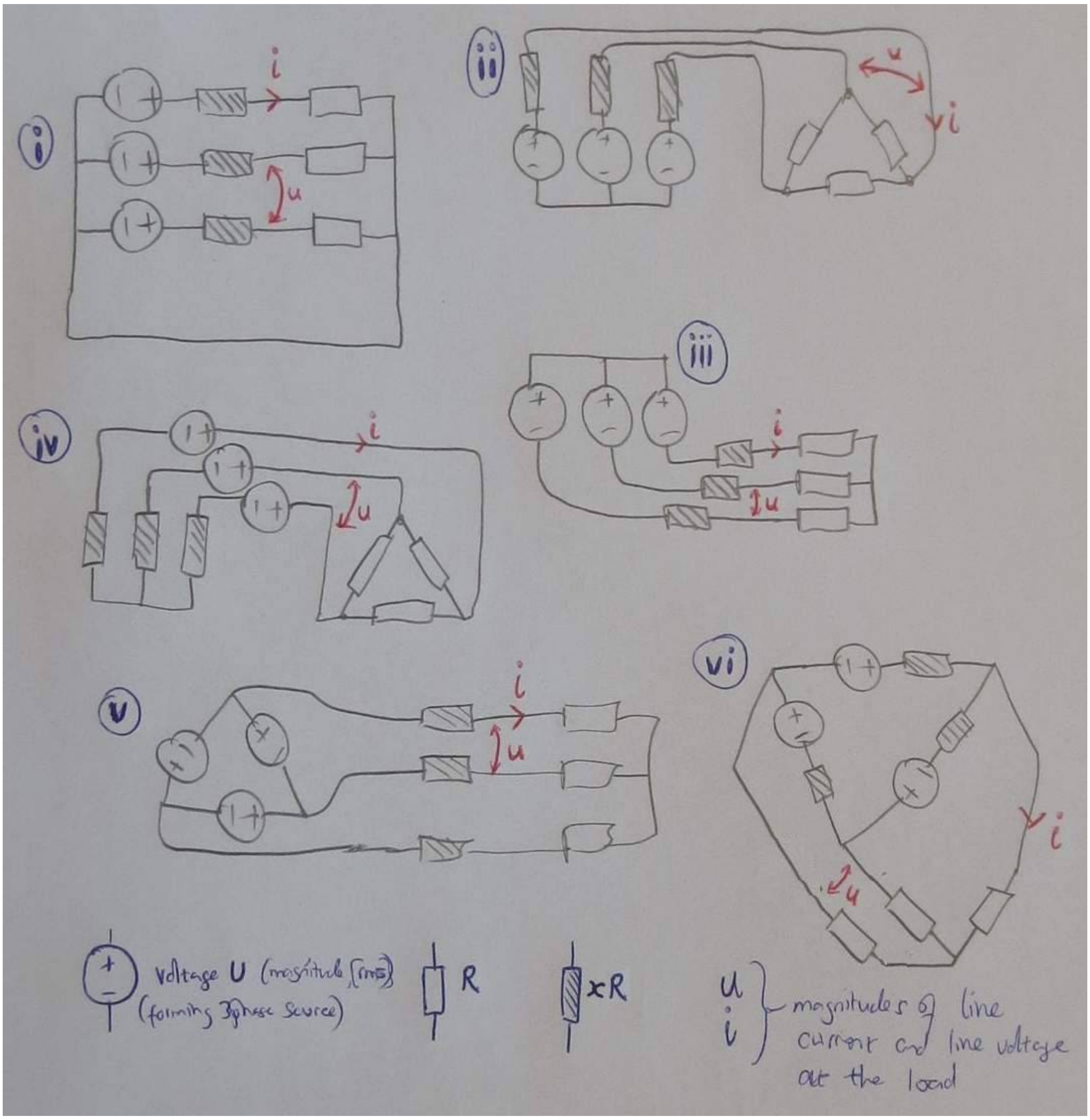
The other quantities are a voltage and current. For the reasons given above, these are identical between the two circuits.

[7] This may *not* be obvious! But, bear in mind that a Δ-connected voltage source only makes sense when the sum of the three single-phase sources is zero: otherwise it would be contradicting KVL. In balanced three-phase conditions this condition of $U_a + U_b + U_c = 0$ is fulfilled; thus, the voltage of any one of the single-phase sources is equal to the voltage that would be found at that point due to the *other* two sources. If it's removed, the three-phase source's terminals still have the same relative voltages. Another way of seeing this is that between three terminals (of the three-phase three-wire source) there can only be two independent voltages, as KVL will determine the remaining third voltage. So these two independent voltage can always be produced by just two sources. In contrast, for a four-wire system (with neutral) one might have to use three sources, as there are three independent voltages between the four wires. The only difference between the first and second diagram is the way that the line currents split between the different single-phase sources: this is a "hidden detail" of the three-phase source, not important for the circuit outside it. We normally draw all three sources in order to imply "balanced three-phase", and because in real applications the sources are typically transformer windings (coils) or generator windings, each one contributing its fair share of the delivered power.

[8] It would be different if there had been two earth nodes in a single circuit-diagram, and they had been changed to be on different nodes between the two diagrams. All earth points shown in a diagram are assumed to be joined together, as part of the same node. There would then be a different *connection* within the circuit, not just a different definition of which potential is called zero. The definition of potential does not affect the key circuit quantities of *voltage*.

Exercise 9

For each circuit, *(i)* to *(vi)*, the impedances with open symbols are resistors of value R, and those with shaded symbols are resistors of value xR, where x is a scaling factor; the voltage sources each have magnitude U, and form a balanced 3-phase set. We can think of the R being a load, and the xR being the impedance of the source and line.



Find, in each circuit: the magnitudes of the marked current i and voltage u; and the total active power supplied to the load resistors, i.e. the three with value R.

(We could have been more general, by using impedances Z and xZ, and asking for complex or apparent power instead of power. But that would divert attention from the core purpose of seeing symmetries and using $\sqrt{3}$ factors.)

Answer 9

(i) This is a classic Y-Y source and load. The shown neutral-connection has no effect, as the system is balanced: no current flows in the neutral, and the voltages in the circuit will not change if the neutral is removed.

The nodes at the left and right (the star-points [neutrals] of the Y-connections) have the same potential, so KVL can be applied for each phase separately: e.g. we can study just the series-connected source and two impedances in the top row, and know that the other rows have the same properties except for having every angle shifted by $\pm 120°$. As we only are interested in magnitudes, we don't even have to keep track of any absolute angle. The current i is found by KVL and Ohm's law. The voltage across one resistor R is found by iR, then this value (the phase-voltage in a Y-connected load) is multiplied by $\sqrt{3}$ to give the line voltage u,

$$i = \frac{U}{(1+x)R} \qquad u = \sqrt{3}iR = \frac{\sqrt{3}U}{1+x} \qquad P = 3i^2R = \frac{3U^2}{(1+x)^2R}.$$

The calculation of power as i^2R was based on seeing that the line current i also flows in the load resistors R. An alternative is to use the standard three-phase power formula $P = \sqrt{3}ui$, as u and i are the line values.

(ii) This is similar, but the load resistors are Δ-connected. We could analyse this directly, by noticing that the current in each resistor R (the load's phase-current) is u/R, and that the line current (for Δ-connection) is therefore $\sqrt{3}u/R$. But we don't directly know u: it is not the same as U or as $\sqrt{3}U$, due to the further resistance xR.

This *can* be handled by solving a suitable pair of equations, but it's easier just to find u and i by Δ-Y conversion of the load. That's the "three-phase short-cut". Then we can analyse just the components in one phase, as in circuit *(i)*. Thus, we convert this circuit to one like *(i)*, but with a load made of resistors $R/3$. Similarly to the previous case, we find u and i,

$$i = \frac{U}{\left({}^1\!/\!{}_3 + x\right)R} = \frac{3U}{(1+3x)R} \qquad u = \sqrt{3}i\frac{R}{3} = \frac{\sqrt{3}U}{1+3x} \qquad P = 3(i/\sqrt{3})^2R = \frac{9U^2}{(1+3x)^2R}.$$

The power here was found by noting that a current $i/\sqrt{3}$ flows in each phase of the Δ load (each resistor R), as this is the relation of line and phase currents for a Δ connection. The same result could be found by analysing the load power in the equivalent Y load, or by the formula $\sqrt{3}ui$. Try!

(iii) Same as circuit *(i)*!

Yes, there's no neutral wire. But we discussed already that that makes no difference: it's all balanced three-phase.

And yes, the sources are "connected funny": their positive-reference sides are on the source's neutral point. That makes no difference: it's as if each source were shifted by 180°, but we didn't define any phase-angles anyway, so that can't matter! (However … if one of the sources had been connected the other way up than the other – e.g. two having the '+' side on the neutral point, and one having the '−' side there – that *would* make a huge difference. Then it wouldn't be a balanced three-phase source any more; it would have 60° shift between the phases [not evenly spaced] and they would sum to $2U$ instead of 0!)

(iv) Same as circuit *(ii)*!

This is getting a little repetitive, isn't it? In each branch (phase) between the neutral point (bottom left) and the connections to the Δ load, there are a source U and a resistor xR, in series. This is true for *(ii)* and for *(iv)*. It doesn't matter which order these two components are arranged in: they are series connected, and KVL along the branch gives the same result either way.

In other words "what the load sees is the same in both cases". This is like our approach to writing nodal analysis equations more simply when we have a branch with several voltage sources and resistors: when we write KCL at the node at the end of the branch, we just look at the sum of resistance and sum of voltage along the branch, as long as we don't care about finding the internal details of potentials *within* the branch.

(v) This looks similar to circuit *(iii)*, except that the *source* is Δ-connected. In both circuits, the source has three terminals, providing a three-phase voltage. What is "seen" at these terminals is identical if this delta-connected source (each phase of which has voltage U) is replaced with a star-connected source in which each phase has voltage $U\sqrt{3}$.

We can therefore (as usual) simplify this circuit to a purely Y-connected one, in which it is clear how each phase can be treated separately. In that case, we can just use the results from *(iii)* or *(i)*, but substitute $U/\sqrt{3}$ instead of U.

$$i = \frac{U}{\sqrt{3}(1+x)R} \qquad u = \sqrt{3}iR = \frac{U}{1+x} \qquad P = 3i^2R = \frac{U^2}{(1+x)^2R}.$$

(vi) Interesting! Redrawing, we see a Y-connected load of resistors R, and a Δ-connection where each phase has a series source U and resistor xR. That's not so familiar. But it's very practically relevant. Realistic voltage sources have some impedance; if we connected them in a delta, this circuit would be a good model.

But we can convert this to look like circuit *(iii)*! Replace the delta-connected phases of U and xR, by a star-connected source $U/\sqrt{3}$, and series resistances $xR/3$! Then the previous solution can be used again, substituting these changes:

$$i = \frac{U}{\sqrt{3}(1+x/3)R} \qquad u = \sqrt{3}iR = \frac{U}{1+x/3} \qquad P = 3i^2R = \frac{U^2}{(1+x/3)^2R}.$$

Is this valid? In circuit *(v)*, a Δ-connected source (with no resistance) was able to be converted to a Y-connected source that gave the same behaviour at its terminals. If the series resistance xR is added to each phase of the Δ-connected source, then a balanced line-current i will cause a phase-current $i/\sqrt{3}$ which passes in each resistor. Thus, the phase-voltage is decreased[9] by $xRi/\sqrt{3}$, and (for a Δ) the line-voltage seen at the terminals changes by the same amount. If we want to try to get the same behaviour from a resistor in series with the Y-connected source in *(iii)*, then we need to make each phase of the source change by $^1/\sqrt{3}$ of this amount, as this is the relation of line and phase voltages in a Y connection. Hence, we want a line-current i to

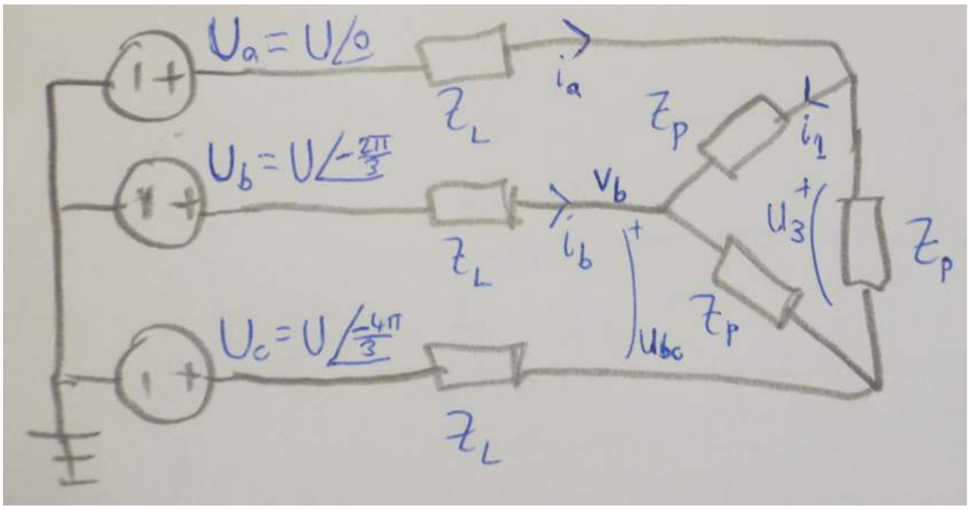
[9]In general, with arbitrary phase of the current (arbitrary load), the phase's total voltage is *changed* by a phasor that depends on the phase-angle of the current. This might reduce or increase the voltage magnitude. But in our case, we know the impedances are all resistors, *and* everything is balanced: so the sources must give output currents that are in phase with their voltages. The voltage drop across the series resistor xR is therefore always exactly subtracting from the source voltage. With capacitors or inductors, or with unbalanced sources, more work would be needed, using phasors all the way.

change the phase voltage by $(xRi/\sqrt{3})/\sqrt{3}$. This is achieved by using a series resistor of $xR/3$, instead of the xR that we had in case *(iii)*, given that the current i flows in this resistor due to the line and phase *currents* being the same in a Y connection.

You are wise if during the above reasoning you are worrying "what about all the phase-shifts between voltage sources, voltage drops across impedances, and currents or voltages in the phases and lines?". We made it simpler having just resistance. For balanced circuits it is valid for arbitrary impedances, as long as their complex values are used throughout the calculation.

Exercise 10

The known quantities in this circuit are U, Z_{L} and Z_{P}.



Find the following quantities as phasors:

a) i_a and i_b

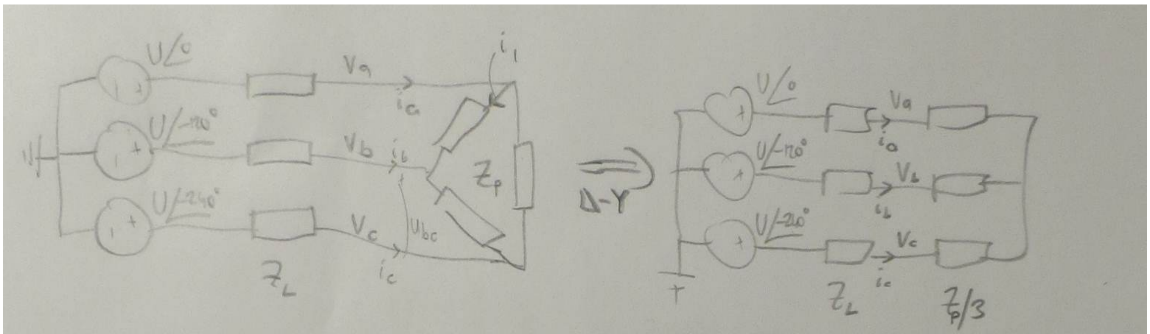
b) i_1

c) u_3

d) u_{bc}

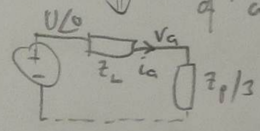
Answer 10

Note also: in answer ‘a)’, $i_b = i_a \cdot \mathrm{e}^{-\mathrm{j}2\pi/3} = (U\underline{/-120^\circ})/(Z_\mathrm{L} + Z_\mathrm{P}/3)$.



Often useful to simplify to single-phase equivalent (valid for BALANCED threephase)

Single phase equivalent of a-phase



(a) $$\boxed{i_a = \frac{U\angle 0}{Z_L + Z_p/3}}$$

see here

can't simplify further when not knowing more detail about Z_L & Z_p

(b) We need to look back at the delta to see "i_1".

One way to find it is as $i_1 = \dfrac{V_a - V_b}{Z_p}$

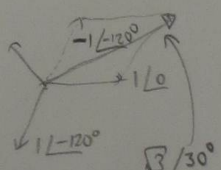
We can find V_a by voltage division in the single-phase diagram

$$V_a = \frac{U\angle 0 \;\; Z_p/3}{Z_p/3 + Z_L} = \frac{U\angle 0 \; Z_p}{Z_p + 3Z_L}$$

V_b is then the same but with -120° phaseshift.

$$\text{So} \quad V_a - V_b = \frac{U Z_p}{Z_p + 3Z_L} \cdot \left(1\angle 0 - 1\angle{-120^\circ}\right)$$

$$= \frac{\sqrt{3}\, U Z_p}{Z_p + 3Z_L} \cdot 1\angle 30^\circ$$



from which $$\boxed{i_1 = \frac{V_a - V_b}{Z_p} = \frac{\sqrt{3}\, U\angle 30^\circ}{Z_p + 3Z_L}}$$

Note this does not mean that $\angle i_1 = 30^\circ$; the values of Z_L & Z_p may be complex.

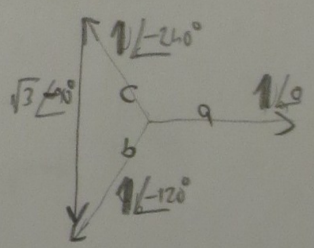
(c) By similar reasoning to the previous,

$$U_3 = V_a - V_c = \frac{U Z_p}{Z_p + 3Z_L} \cdot \left(1\angle 0 - 1\angle -240^\circ\right)$$

$$\boxed{\therefore U_3 = \frac{\sqrt{3}\, U Z_p}{Z_p + 3Z_L} \cdot 1\angle -30^\circ = \frac{\sqrt{3}\, Z_p\, U\angle -30^\circ}{Z_p + 3Z_L}}$$

(d) and $U_{bc} = V_b - V_c = \dfrac{U Z_p}{Z_p + 3Z_L}\left(1\angle -120^\circ - 1\angle -240^\circ\right)$

$$\boxed{\therefore U_{bc} = \frac{\sqrt{3}\, Z_p\, U\angle -90^\circ}{Z_p + 3Z_L}}$$



If the load had been directly connected to the source $(Z_L = 0)$ then the only difference would be that the $\dfrac{Z_p}{Z_p + 3Z_L}$ term would not be found in b) c) e d).

Exercise 11

A modern aeroplane's "Auxiliary Power Unit" (APU), i.e. an on-board generator, can be used to provide lighting, air-conditioning and engine-starting while the plane is parked. When available, a "Ground Power Unit" (GPU) is preferred for supplying the required electricity for a parked aircraft, in order to avoid the fuel-burn and noise of an APU. The energy can then be obtained from the airport's power network, passing through a power-electronic converter that provides the necessary voltage and frequency for the plane. You might have seen the fairly thick multi-wire cables for ground power being plugged into a plane that has landed.

One example of a GPU is the Hobart 2400 `exampleGPU[pdf]` which converts ground power to the required voltage and frequency, and supplies this to the plane through a 24 m cable. From a Boeing 787 specification acaps-787[pdf] (p.67–72), we see that this plane has inlets for up to three GPU cables of 90 kVA, 200/115 V (three-phase), 400 Hz, with all three being needed in order to start the engines in the worst conditions while also keeping all cabin loads running. This is a quite modern plane that is "all electric"; others may require pneumatic connections to run some services.

The voltage specification means 200 V line voltage (huvudspänning), and therefore $200\,\mathrm{V}/\sqrt{3} = 115\,\mathrm{V}$ as the phase voltage (fasspänning) between each line and the neutral. It's a common way of showing both voltages, and thereby implying a "three-phase four-wire" system. The low-voltage distribution network that we're used to in Europe is often described as '400/230 V, 50 Hz', or sometimes with the voltages reversed: 230/400 V.

The frequency 400 Hz is common in aircraft and some ships. An advantage of higher frequency is that electromagnetic machines (generators, motors, transformers) can have a smaller size and mass for a given power rating when the frequency is higher. A disadvantage of higher frequency is that inductive reactances are increased (giving more voltage drop in the lines when load current passes through them) and capacitive reactances are decreased (requiring more current to charge the parallel capacitance of cables).

a) What current does the 90 kVA GPU described above have to supply when providing its full rated power? This is the current magnitude in each of the three conductors, i.e. the line current; people dealing with networks might say "the current in each phase of the cable".

b) Suppose that the 24 m cable between the GPU output and the plane's input has a series resistance $R = 10\,\mathrm{m\Omega}$ and series inductance $L = 7.0\,\mathrm{\mu H}$ per phase.[10] Consider the situation where the voltage at the plane is the rated value of 200 V, and the power flow into the plane at this point is 90 kVA at PF = 1.
What is the line-voltage magnitude at the source (GPU) end of the cable?
What reactive power does the source (GPU) supply?

c) Do subquestion 'b)' again, but with PF = 0.5 lagging.

[10] No details about the GPU cable have been found. The resistance suggested here is realistic for a cable that can carry the current we find in question 'a)', as that current suggests rather more than $50\,\mathrm{mm}^2$ of copper being needed. The inductance suggested here is typical for a cable where the conductors are tightly bundled together: it would be more if they were far apart as in an open-wire overhead line.

Answer 11

a) Using total complex power S, and [rms] line-quantities U and I,

$$|I| \;=\; \frac{|S|}{\sqrt{3}|U|} \;=\; \frac{90.0\,\mathrm{kVA}}{\sqrt{3}\cdot 200\,\mathrm{V}} = 260\,\mathrm{A}.$$

b) Each phase of the line (cable) has an impedance of

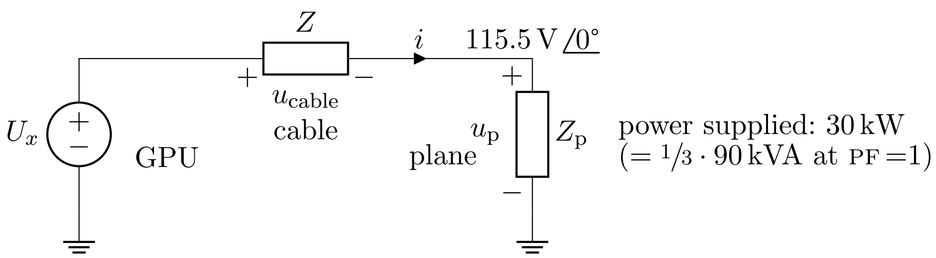
$$Z \;=\; R + \mathrm{j}\omega L \;=\; 10.0\,\mathrm{m\Omega} + \mathrm{j}2\pi\cdot 400\,\mathrm{Hz}\cdot 7.0\,\mathrm{\mu H} \;=\; (10.0{+}\mathrm{j}17.6)\,\mathrm{m\Omega}.$$

The magnitude of this impedance is close to $20\,\mathrm{m\Omega}$. When the line current of $260\,\mathrm{A}$ passes through this impedance, the magnitude of the voltage across this phase of the cable is

$$|u_{\text{cable}}| = |I||Z| = 260\,\mathrm{A}\cdot 20\,\mathrm{m\Omega} = 5.3\,\mathrm{V}.$$

But we *cannot* just add this (as a scalar) to the voltage at the plane, to find the voltage at the GPU. In our KVL we have to consider the phasors, with *angle* as well as magnitude; the angles of the impedance and of the load-current (relative to the load voltage) are both relevant.

Probably the easiest approach is to consider just a single phase of this balanced three-phase circuit, treating all loads and sources as Y-connected. In a balanced circuit, we know that all three phases have identical diagram, powers, impedances, and voltage and current magnitudes; the difference is just that all voltage and current angles are shifted 120° between the phases. Let's draw one phase, and define the voltage u_{p} at the plane as our reference angle. This will be the phase voltage, of $\frac{200\,\mathrm{V}}{\sqrt{3}} = 115.5\,\mathrm{V}$.[11] The power supplied in this phase is 1/3 of the total power.



The load's complex power and phasor voltage are known, from which the phasor line-current is

$$i = \left(\frac{(30{+}\mathrm{j}0)\,\mathrm{kVA}}{115.5\,\mathrm{V}}\right)^{*} = (260{+}\mathrm{j}0)\,\mathrm{A},$$

which is purely real, as the load's power is purely real (active power) and its voltage was defined as the reference angle.

The voltage marked across this phase-conductor of the cable is

$$u_{\text{cable}} = iZ = (260{+}\mathrm{j}0)\,\mathrm{A}\cdot(10.0{+}\mathrm{j}17.6)\,\mathrm{m\Omega} = (2.6{+}\mathrm{j}4.6)\,\mathrm{V}.$$

[11] The specification of '200/115' rounds the phase voltage to 115 V, but $\frac{200\,\mathrm{V}}{\sqrt{3}}$ is much closer to 115.5 V. We will keep our calculation a bit more precise than is of practical need, just so that we don't get confused later by small differences between the phase and line voltages in our final answer.

The magnitude of voltage U_x at the source is

$$|U_x| = |\, u_\mathrm{p} + u_\mathrm{cable}\,| = |\,(115.5{+}\mathrm{j}0)\,\mathrm{V} + (2.6{+}\mathrm{j}4.6)\,\mathrm{V}\,| = |\,(118.1{+}\mathrm{j}4.6)\,\mathrm{V}\,| = 118.2\,\mathrm{V}.$$

Notice that the imaginary part of the cable's voltage had very little effect on the difference in voltage magnitudes between the two ends of the cable, because it was added at 90° to a much bigger number: think of 'small angle approximation', or that $\sqrt{1+k^2} \simeq 1$ if $k \ll 1$.

Final step! Remember that we've studied just one phase, considering loads and sources as being single phases of a Y-connected three-phase system. If the phase-voltage is larger by $118.2\,\mathrm{V} - 115.5\,\mathrm{V} = 2.7\,\mathrm{V}$ at the source than the load, then in a balanced system the line-voltage should be larger by $\sqrt{3} \cdot 2.7\,\mathrm{V} = 4.7\,\mathrm{V}$. Alternatively, we can calculate the line voltage at the source directly from the phase voltage 118.2 V at the source,

$$U_{\mathrm{GPU}} = \sqrt{3} \cdot 118.2\,\mathrm{V} = 204.7\,\mathrm{V}.$$

c) Repeat 'b)', but PF = 0.5 lagging. Easily done, following the exact procedure used above, but with a different complex power at the load.

The power-factor angle is $\theta = \cos^{-1}(0.5) = 60°$. This is the angle between the voltage and current phasors. It represents the current lagging the voltage, because we were told that the power factor was lagging, and voltage is the conventional reference angle for defining lagging or leading power factors. So the complex power is

$$S = |S| \cdot \mathrm{e}^{\mathrm{j}\pi/3} = (45.0{+}\mathrm{j}77.9)\,\mathrm{kVA},$$

and the line current is

$$i = \left(\frac{\frac{1}{3} \cdot (45.0{+}\mathrm{j}77.9)\,\mathrm{kVA}}{115\,\mathrm{V}} \right)^{*} = (130{-}\mathrm{j}225)\,\mathrm{A},$$

leading to a voltage across the cable of

$$u_\mathrm{cable} = iZ = (130{-}\mathrm{j}225)\,\mathrm{A} \cdot (10.0{+}\mathrm{j}17.6)\,\mathrm{m\Omega} = (5.3{+}\mathrm{j}0.035)\,\mathrm{V}.$$

The magnitude of voltage U_x at the source is

$$|U_x| = |\,(115.5{+}\mathrm{j}0)\,\mathrm{V} + (5.3{+}\mathrm{j}0.035)\,\mathrm{V}\,| = 120.8\,\mathrm{V},$$

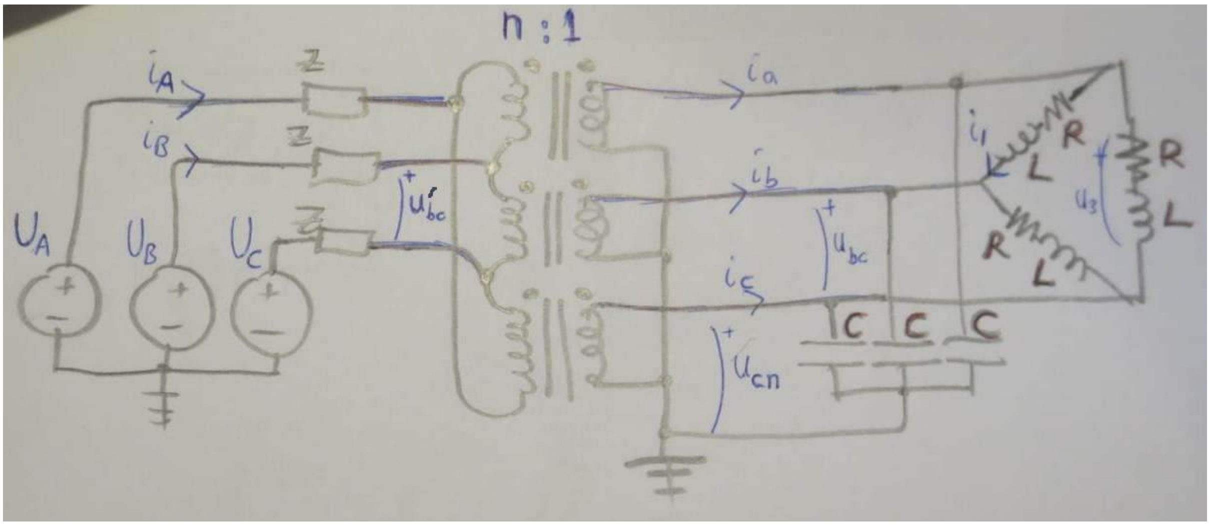
so if we think back to the actual circuit (not just the single-phase model), the line voltage at the GPU is

$$U_{\mathrm{GPU}} = \sqrt{3} \cdot 120.8\,\mathrm{V} = 209.2\,\mathrm{V}.$$

With this lower power factor, we have seen *more* difference in voltage between the sending and receiving ends of the cable. That is because the load current angle (relative to load voltage) is now very similar to the angle of the line's impedance. The voltage u_cable is then almost in phase with the load voltage, so the phasor sum of the two voltages is almost the same as the sum of magnitudes.

Exercise 12

This diagram shows a three-phase source, coupled through a line of impedance Z to the primary of a three-phase transformer that is made of three ideal single-phase transformers. The secondary side of the transformer feeds a Δ-connected inductive-resistive load and a set of 'power factor correction' capacitors.



The line voltage at the transformer primary is 10 kV, so $|u'_{bc}| = 10\,\text{kV}$.
The frequency is 50 Hz.
The transformer ratio is $n = 25$.
Use the source's a-phase (U_A) as the reference angle: $\angle U_A = 0$.

a) Find $|u_{cn}|$, $|u_{bc}|$ and $|u_3|$.

b) Find $|i_1|$ and $|i_a|$ in terms of R, L and C. For neatness, define $\omega = 2\pi \cdot 50\,\text{Hz}$ and $u = |u_{cn}|$.

c) What reactive power Q and apparent power $|S|$ does the Δ-connected load consume, if its active power consumption is 30 kW and its power-factor is 0.9 lagging?

d) What values do R and L have if this Δ-connected consumes the complex power indicated in subquestion 'c'?

e) What value must C then be in order to give complete power-factor correction, so that the transformer supplies only active power?

f) What are the magnitudes of line-currents $|i_a|$ and $|i_A|$?

g) If $Z = (3.16 + \text{j}3.16)\,\Omega$, what is $|U_A|$?

h) Again assuming $Z = (3.16 + \text{j}3.16)\,\Omega$, what is u_{cn}?
Note: *not* just as magnitude … phase-angle also.

Answer 12

a) The transformer's windings are in Δ (primary) – Y (secondary) connection.
Everything's balanced in this system.
We are told the line-voltage on the primary side of the transformer: it is $|u'_{bc}| = 10\,\mathrm{kV}$.
Each single-phase transformer has its primary connected between two lines, i.e. 10 kV.
With the transformer's ratio $n = 25$, the voltage on its secondary side is $10\,\mathrm{kV}/25 = 400\,\mathrm{V}$.
The marked u_{cn} is the voltage across one of these secondary windings, so $|u_{cn}| = 400\,\mathrm{V}$.
The marked line-voltage u_{bc} on the secondary side is $|u_{bc}| = \sqrt{3} \cdot 400\,\mathrm{V} = 692\,\mathrm{V}$, as the secondary sides of the transformers are Y-connected and have a phase-voltage of 400 V.
Voltage u_3 is also $|u_3| = 692\,\mathrm{V}$, as this voltage is marked between two lines.

b) Current i_1 is defined in one phase of the Δ-connected balanced load. The voltage across this phase has magnitude 692 V, as we see from subquestion 'a'. By the relation

$$\text{current magnitude} = \frac{\text{voltage magnitude}}{\text{impedance magnitude}},$$

we find

$$|i_1| = \frac{692\,\mathrm{V}}{\sqrt{R^2 + \omega^2 L^2}}.$$

Current i_a is the line-current to this Δ load, added to the current to the Y-connected capacitors. This addition has to be done using phasors, even if we only want magnitude in our final result: the magnitude of the total current depends on the relative angles of the added currents. It seems simplest and safest to do a Δ-Y conversion so that each phase of the Y-connected equivalent is a series $R/3$ and $L/3$, and then to consider just one phase. We can start by finding the total admittance of the C and L-R loads (per phase), and simplifying this. Then the magnitude of current can be found from the magnitude of voltage and the magnitude of admittance, by the same principle as we used for $|i_1|$. Alternatively, we could work with currents from the start, which would just mean that an extra factor of voltage 400 V at some arbitrary angle would have to be carried through the equations. The admittance of each phase is

$$Y \;=\; \frac{1}{\frac{R}{3} + \mathrm{j}\omega\frac{L}{3}} + \mathrm{j}\omega C \;=\; \frac{3R}{R^2 + \omega^2 L^2} + \mathrm{j}\left(\omega C - \frac{3\omega L}{R^2 + \omega^2 L^2}\right),$$

which has magnitude

$$|Y| = \frac{\sqrt{(3R)^2 + \left(\omega C R^2 + \omega^3 C L^2 - 3\omega L\right)^2}}{R^2 + \omega^2 L^2}.$$

When the phase-voltage of 400 V is applied to this admittance, the current into it will have the same magnitude as the sought line-current $|i_a|$,

$$|i_a| = 400\,\mathrm{V} \cdot |Y|.$$

There seems little point copying all the long expression for $|Y|$ into this equation: we know how we could calculate if we had values of the components.

c) What reactive power Q and apparent power $|S|$ does the Δ-connected load consume, if its active power consumption is 30 kW and its power-factor is 0.9 lagging?

From PF $= 0.9$ we know

$$|S| = \frac{P}{0.9} = \frac{30.0\,\mathrm{kW}}{0.9} = 33.3\,\mathrm{kVA},$$

and from this,

$$Q = \sqrt{|S|^2 - P^2} = 14.5\,\mathrm{kvar}.$$

We know that this positive value of reactive power is the value *into* the load; reactive power is consumed by a load that has lagging power factor. We can confirm the reasonableness of lagging power factor, since this load consists only of resistance and inductance.

d) What values do R and L have if P and Q are the values in 'c', above.

For an impedance Z, with voltage magnitude u applied to it,

$$S = \frac{|u|^2}{Z^*}.$$

In our case, $S = (1/3) \cdot (33.3{+}\mathrm{j}14.5)\,\mathrm{kVA}$ because the impedance in each phase of the load contributes $1/3$ of the load power. The voltage across each is $u = 692\,\mathrm{V}$. The impedance for each series R-L pair is then

$$Z = \frac{|u|^2}{S^*} = \frac{(692\,\mathrm{V})^2}{\frac{1}{3} \cdot (30.0{-}\mathrm{j}14.5)\,\mathrm{kVA}} = (38.8{+}\mathrm{j}18.8)\,\Omega = R + \mathrm{j}\omega L.$$

By equating real and imaginary terms in the above, we get

$$R = 38.8\,\Omega \qquad \text{and} \qquad L = \frac{18.8\,\Omega}{2\pi \cdot 50\,\mathrm{Hz}} = 59.7\,\mathrm{mH}.$$

e) With all the above values of L, u_{cn} etc, what value of C is needed in order to give complete power-factor correction of the Δ-connected load?

Looking back to subquestion 'b', we have an expression for the total admittance Y per phase. Or if we'd chosen an alternative method, we might have found an expression for the total current caused by applying 400 V to this admittance.

These expressions apply to the combination of the Y-connected load and the capacitors. Complete power-factor correction means that this combination has unity power-factor, or in other words has zero reactive power: the capacitors supply what reactive power the load consumes.

From the expression for Y, this condition requires

$$\omega C - \frac{3\omega L}{R^2 + \omega^2 L^2} = 0 \qquad \Longrightarrow \qquad C = \frac{3L}{R^2 + \omega^2 L^2}.$$

Putting in the values we found before,

$$C = \frac{3L}{R^2 + \omega^2 L^2} = \frac{3 \cdot 59.7\,\mathrm{mH}}{(38.8\,\Omega)^2 + (2\pi \cdot 50\,\mathrm{Hz} \cdot 59.7\,\mathrm{mH})^2} = 96\,\mu\mathrm{F}.$$

f) Line-current magnitudes $|i_a|$ and $|i_A|$, given the values of R, L, C determined in earlier subquestions.

The secondary-side line-current magnitude $|i_a|$ was already derived in subquestion 'b', in terms of R, L, C, etc. Now that we know these values we can find $|i_a|$; we can even try being cunning, noticing that the imaginary part in the expression for Y can be ignored (since we chose C to set this part to zero), so

$$Y = \frac{3R}{R^2 + \omega^2 L^2}.$$

The actual line current is identical to the phase-current in the single-phase Y-equivalent model for which Y was derived, so

$$|i_a| = |400\,\mathrm{V}| \cdot \left| \frac{3R}{R^2 + \omega^2 L^2} \right| = \text{etc!}$$

But we might notice that there's a quicker way! The load consumes active power of 30 kW. With complete power-factor compensation of the load, the transformer therefore supplies $S = P = 30\,\mathrm{kW}$. So, by the familiar formula for balanced three-phase power in terms of line voltage and current,

$$|i_a| = \frac{30\,\mathrm{kW}}{\sqrt{3} \cdot 692\,\mathrm{V}} = 25.0\,\mathrm{A}.$$

To find the line-current magnitude at the primary side of the transformer, $|i_A|$, we can do the same: we know the line voltage at the transformer's primary, and we know that an ideal transformer must have the same complex power going in as coming out,

$$|i_A| = \frac{30\,\mathrm{kW}}{\sqrt{3} \cdot 10\,\mathrm{kV}} = 1.73\,\mathrm{A} \qquad \text{Yep: genuinely } \sqrt{3}\,\mathrm{A!}$$

Another way to find this would be to note that the secondary-winding phase currents are equal to the secondary line currents (Y-connection); the primary-winding phase currents are $1/n$ of this; then the primary line-currents are a factor $\sqrt{3}$ larger (due to Δ-connection):

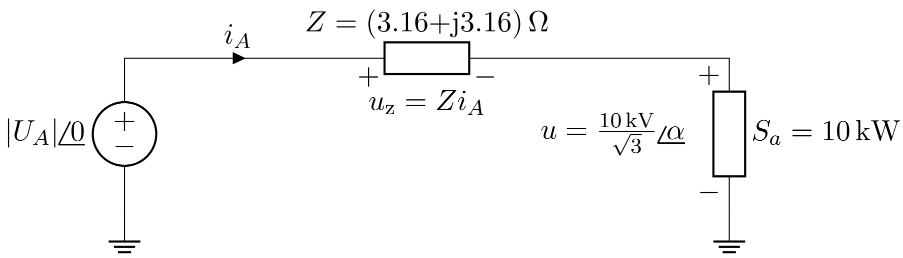
$$|i_A| = \frac{\sqrt{3}}{n} |i_a| = 1.73\,\mathrm{A}.$$

g) $|U_A|$ given that $Z = (3.16\mathrm{+j}3.16)\,\Omega$.

As usual for balanced conditions, it's convenient to use a per-phase view of the circuit.

The load has complete power-factor correction, and the transformer is ideal, so the power transfer at the transformer primary is purely real. It is 30 kW, which means 10 kW per phase.

We know the line-voltage magnitude at the transformer primary is $|u'_{ab}| = 10\,\mathrm{kV}$. For our per-phase equivalent we can consider phase-a; this is convenient, as phase-a of the source is the reference angle. Let's define as u the phase-a potential at the transformer, denoting its unknown angle as α. The per-phase equivalent is then the following:



In this circuit, the variables we already know are Z, $|i_A|$, and $|u|$.
We need to find $|U_A|$. The obvious way to start is KVL,

$$U_A = u + u_{\mathrm{z}} = u + Z i_A \qquad \text{or} \qquad |U_A|\angle 0 = \frac{10\,\mathrm{kV}}{\sqrt{3}}\angle\alpha + (3.16\mathrm{+j}3.16)\,\Omega \cdot |i_A|\angle\beta \quad (\text{where } \beta = \angle i_A),$$

but this equation contains *three* unknown parts: the magnitude $|U_A|$, the angle α and the angle β of the current i_A. The circuit is not directly solvable from just this information; to find $|U_A|$ the phasor sum of $u + u_z$ is needed, but both of these voltages have unknown angles.

However, to find the magnitude of the phasor sum, we only need to know the *relative angles* of u and u_z. This, fortunately, can be found from our knowledge that $S_a = 10\,\text{kW}$. This tells us that

$$i_A = \left(\frac{S_a}{u}\right)^*,$$

$$\angle i_A = -(\angle S_a - \angle u) \qquad \Longrightarrow \qquad \angle u - \angle i_A = \angle S_a = \angle(10{+}\text{j}0)\,\text{kVA} = 0.$$

In other words, as the load looks purely real ('resistive'), the current i_A and voltage u are in phase with each other. The magnitude $|u + u_z|$ is then

$$|U_A| = |u + u_z| = \Big||u| + |i_A|Z\Big| = \left|\frac{10\,\text{kV}}{\sqrt{3}} + 1.73\,\text{A} \cdot (3.16{+}\text{j}3.16)\,\Omega\right| = 5.78\,\text{kV}.$$

Notice that the same result would have come if instead of taking magnitudes of $|u|$ and $|i_A|$ we had given both of these quantities an arbitrary angle, the same for both. If S_a had not been purely real, then we'd have to have added a further angle of $\angle S_a$ to u in the calculation.

This result for $|U_A|$ is very similar to the voltage magnitude $|u|$, since u_z is so small: the voltage difference is 5.5 V in 5.77 kV.

We have now got the solution, $|U_A| = 5.78\,\text{kV}$. This quantity was part of the per-phase circuit drawn above: that's because U_A is one phase of a Y-connected source, and we chose to work with phase-a. If instead the line-voltage at the source had been requested, it would be necessary to multiply by $\sqrt{3}$.

h) u_{cn} given $Z = (3.16{+}\text{j}3.16)\,\Omega$.

As seen in part 'g', the voltage across impedance Z is very small compared to U_A, u'_{bc} etc. There is negligible difference in angle, between the source and the transformer primary.[12] The zero angle defined in this system is phase-a of the source. The bottom transformer supplies the voltage u_{cn} on its secondary, seen from the dotted end relative to the other end. The primary, seen in the same direction with respect to the dots, sees 10 kV with a phase angle of $\angle U_C - U_A = 150°$, on the above assumption that there is negligible phase-shift from the currents $i_{A,B,C}$ in the impedances Z.

Therefore, $u_{cn} = (400\angle 150°)\,\text{V}$, or $u_{cn} = (-346{+}\text{j}200)\,\text{V}$.

[12]If you sensibly want to know "how negligible", then the result of `(180/pi) * angle( 1.73*(3.16+3.16j) + 10e3/sqrt(3) )` tells us the angle is 0.05°!