KTH, Electric Circuit Analysis, EI1120

Topic 14 Practice Exercises

Calculations in UNBALANCED Three-Phase Circuits

Now we come to three-phase systems where there is some unbalance, usually in the loads.

This is common in practice, especially in the 'low-voltage' networks that supply individual homes and other buildings. Many of the loads are single-phase, connected from a line to neutral. Unless by good luck the loads connected to the three phases are very similar, the system cannot be treated as a balanced three-phase system.

Much of the symmetry that has been useful in our earlier calculations is unfortunately not directly applicable here. Likewise, some of the 'standard equations' such as three-phase power based on $\sqrt{3}u_{\mathrm{L}}i_{\mathrm{L}}$ are not applicable ... which current or voltage should one choose for this formula, if they could be different in each line? However, it is sometimes possible to take advantage of symmetry by first separating the system into a balanced system plus some extra components that add the unbalance. Then one can hope that the balanced part can simplify a lot, after which the unbalanced parts are added and a relatively easy solution obtained. Unfortunately, in some unbalanced cases we really just have to calculate the system fully, with careful attention to all the different phase-angles etc. That's not so hard in reality, when using computers – users in industry will normally have programs that someone else has written, to calculate on a system that is specified by clicking transformers, lines, loads, etc, into a diagram. But when we do it by hand in this course, we keep our circuits for unbalanced calculations conveniently small.

As with the exercises on balanced three-phase calculations, the following exercises vary from quite easy up to distinctly long and difficult. Hardly anyone is expected to go beyond about half way. The rest are for the hypothetical few who revel in this sort of work; I wrote some of them for Elektro last year.

Here are some further exam questions with unbalances.

2016-06˙EM˙omtenta Q9-bc single R and C, as unbalanced load

2016-03˙EM˙tenta Q9-c unbalanced R-load

2013-03˙EM˙tenta Q7 unbalanced load and transformer

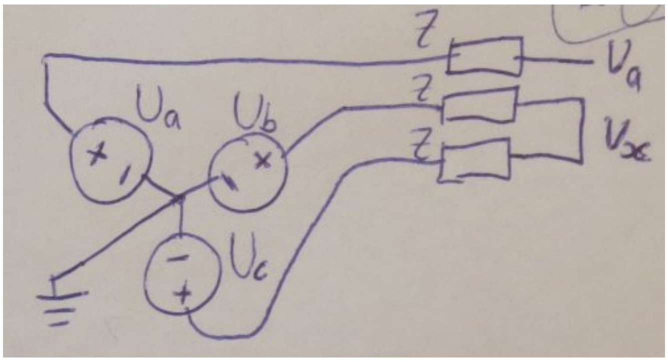
2013-06˙EM˙omtenta Q7 example with unbalanced load

2013-06˙E˙omtenta Q5 balanced and unbalanced situations

2015-06˙EM˙omtenta Q9-c a little unbalance

Exercise 1

Predictably, the sources here produce a three-phase voltage with line-voltage magnitude $\sqrt{3}U$. Phase-rotation is a,b,c (which we can assume by default).
Define the reference angle as the source's a-phase, so that $U_a = U\underline{/0}$.
We'll use the star-point of the source as the reference potential, as shown.

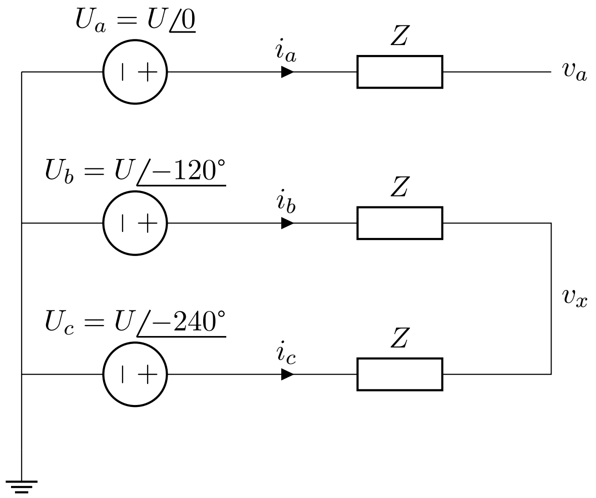


Find the marked potentials, v_a and v_x, as phasors.

Practical aspect: the value of v_x in this case has a relevance to the case of a short-circuit fault between two conductors of the line (with Z representing the impedance of each phase of the source and line up to the point of that fault). Then v_x is the voltage with respect to the ground.

Answer 1

This is another unbalanced circuit, but it's not very much work to solve. (It's unbalanced because of the connections. The source is balanced. The three impedances are identical so they *could* form a balanced three-phase line or load. But they're not connected in a balanced three-phase way: it's not symmetric between the phases.)



The end of the impedance Z where v_a is defined is open, meaning $i_a = 0$.
Therefore, by Ohm's law and KVL, $v_a = U_a = U\underline{/0}$.

We can define v_x by KVL, by using the marked but unknown currents,

$$v_x = U_c - Zi_c \qquad \text{or} \qquad v_x = U_b - Zi_b.$$

Looking around the loop of U_b, Z, Z, U_c, these currents are

$$i_b = -i_c = \frac{U_b - U_c}{2Z}.$$

Putting these together,

$$v_x = U_c - Z\frac{-(U_b - U_c)}{2Z} = \frac{U_b + U_c}{2}.$$

The given quantities were U (and Z), so we substitute,

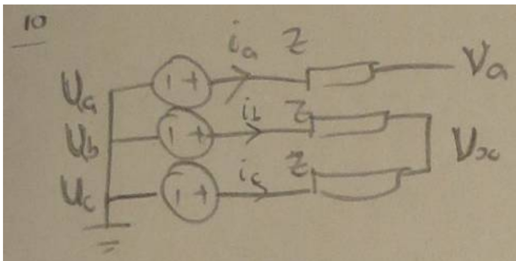
$$v_x = \frac{U}{2}\left(1\underline{/-120°} + 1\underline{/-240°}\right) = \frac{U}{2}\underline{/-180°} = \frac{-U}{2}.$$

So, $v_x = -U/2$.

This is independent of the value of Z, since both phases have the same impedance Z. In a phasor diagram we would see that the potential v_x is a point on the line between U_b and U_c.

We should not be surprised that $1\underline{/-120°} + 1\underline{/-240°}$ became -1, as we know that a symmetric three-phase set of phasors can be described as $\{\ 1\underline{/0},\ 1\underline{/-120°},\ 1\underline{/-240°}\ \}$, and that these three phasors sum to zero.

Answer 1 (Alternative form)

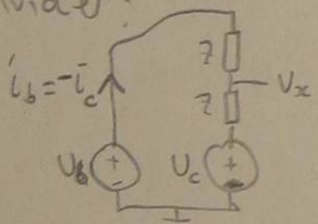


$U_a = U\angle 0$

This is unbalanced. But it's not too bad.

- $i_a = 0$, as nothing is connected at the end.
  So $i_a Z = 0$, meaning $\boxed{V_a = U_a = U\angle 0}$

- Looking at the connection around V_x, we can see V_x as being the middle of a voltage divider:

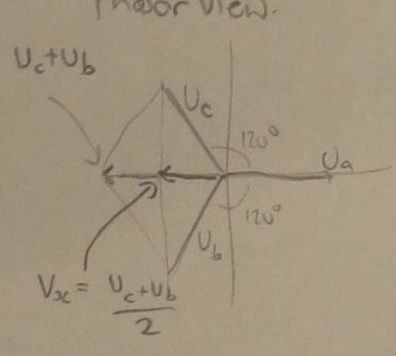


$$\frac{U_b - V_x}{Z} + \frac{U_c - V_x}{Z} = 0 \quad \text{(KCL)}$$

$$\Rightarrow V_x = \frac{U_b + U_c}{2} = \frac{U}{2}\left(1\angle{-120^\circ} + 1\angle{-240^\circ}\right)$$

$$\therefore \boxed{V_x = \frac{-U}{2} = \frac{U}{2}\angle \pi}$$

Phasor view:



We should not be surprised that

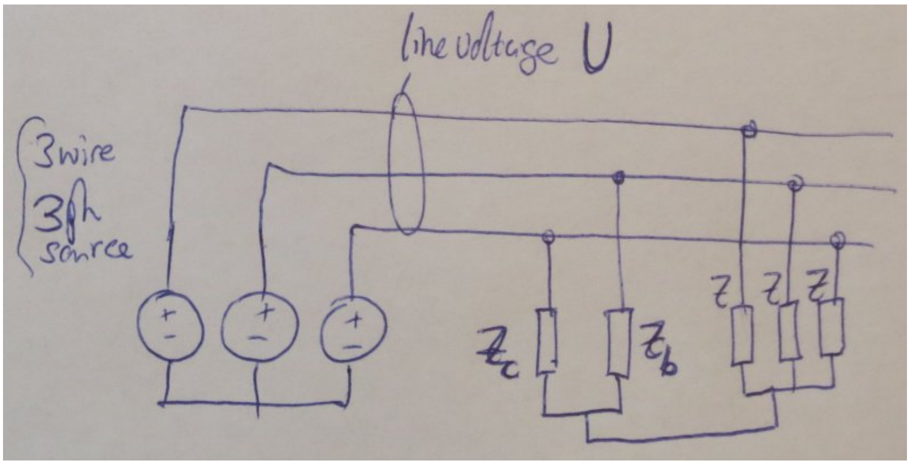
$1\angle{-120^\circ} + 1\angle{-240^\circ} = -1$,

because we know that

$1\angle 0^\circ + 1\angle{-120^\circ} + 1\angle{-240^\circ} = 0$

Exercise 2

This is a balanced load and an unbalanced load, connected to a balanced three-phase source. The two loads have a common neutral point, but this has no connection to the source neutral.



a) Suppose that all the impedances shown here are similar resistors of resistance R:
$Z_c = Z_b = Z = R$.
How much power is dissipated in Z_c in this case?

b) Now suppose that $Z_c = Z_b = R$, but that $Z = \frac{-\mathrm{j}}{\omega C}$, where $\frac{1}{R} = \omega C$.
How much power is dissipated in Z_c now?

Answer 2

a) How much power is dissipated in Z_c if $Z_c = Z_b = Z = R$?

Whether by hand or by computer, a sensible-looking approach is by nodal analysis, at the single node of the neutral point of the impedances Z: let us call this v_x. Let's choose the neutral of the *source* to be the reference node (ground): then v_x is the only unknown potential, so

$$\frac{v_x - U_a}{Z} + \frac{v_x - U_b}{Z} + \frac{v_x - U_c}{Z} + \frac{v_x - U_c}{Z_c} + \frac{v_x - U_b}{Z_b} = 0$$

where U_a is the phasor of the voltage-source at the left, which we can define as $U_a = \frac{U}{\sqrt{3}}\angle 0$ (we are free to choose any reference angle we want, since we only need to find a voltage or current magnitude in order to determine the power in Z_c). The other two sources U_b and U_c have the further phase-delays expected for a three-phase source (balanced, phase-rotation a,b,c). Here, we've expressed the impedances as separate symbols instead of just R, in order to use the same equation later.

Due to the balanced three-phase source, we know $U_a/Z + U_b/Z + U_c/Z = 0$, so

$$\frac{3v_x}{Z} + \frac{v_x}{Z_c} + \frac{v_x}{Z_b} = \frac{U_c}{Z_c} + \frac{U_b}{Z_b}$$

whence

$$v_x = \frac{\frac{U_c}{Z_c} + \frac{U_b}{Z_b}}{\frac{3}{Z} + \frac{1}{Z_c} + \frac{1}{Z_b}}.$$

In our special case, where $Z_c = Z_b = Z = R$,

$$v_x = \frac{U_c + U_b}{5}.$$

The power delivered to the impedance Z_c (which is a resistance R) is

$$P_c = \frac{\left|U_c - \frac{U_c+U_b}{5}\right|^2}{R} = \frac{\left|\frac{4}{5}U_c - \frac{1}{5}U_b\right|^2}{R} = \frac{\left|\frac{4}{5}\frac{U}{\sqrt{3}}\angle{-240^\circ} - \frac{1}{5}\frac{U}{\sqrt{3}}\angle{-120^\circ}\right|^2}{R}.$$

By inserting $\cos\frac{-2\pi}{3}$ and $\sin\frac{-2\pi}{3}$ to get rectangular components in the above expressions,

$$P_c = \frac{\left(\frac{U}{\sqrt{3}}\right)^2 \left(\left(\frac{-1}{2}\left(\frac{4}{5} - \frac{1}{5}\right)\right)^2 + \left(\frac{\sqrt{3}}{2}\left(\frac{4}{5} - \frac{-1}{5}\right)\right)^2\right)}{R} = \frac{U^2}{3R}\left(\frac{9}{100} + \frac{3}{4}\right) = \frac{\left(\frac{\sqrt{7}}{5}\right)^2 U^2}{R}.$$

What if we actually cared about the answer? (Practically.)
Then we'd be sensible do it in Octave/Matlab/etc — quick, simple and reliable — instead of making mistakes manipulating diagrams and symbols. The main advantage of the symbolic way is to find a general-case equation in some elegant form that helps us see what effects all the variables have.

The numerical calculation, however, assumes we have some values for the circuit quantities U and R, Let's choose $U = 400\,\text{V}$ and $R = 15.87\,\Omega$.

```
% rather longer way (for general impedances)
U = 400;  R = 15.87;
Ua = U/sqrt(3);
Ub = U/sqrt(3) * exp(-1j*2*pi/3);
Uc = U/sqrt(3) * exp(-1j*4*pi/3);
Zc = R;   Zb = R;   Z = R;
vx = ( Uc/Zc + Ub/Zb ) / (1/Zc + 1/Zb + 3/Z)
Pzc = real( abs(Uc-vx)^2 / conj(Zc) )     % becomes:  2822.9
```

```
% short way (when known that all impedances are R)
U = 400;  R = 15.87;
s = exp(-1j*2*pi/3);
Ua = U/sqrt(3);  Ub=s*Ua;  Uc=s*Ub;
vx = ( Uc + Ub ) / 5
%  solve for the requested power:
Pzc = abs(Uc-vx)^2 / R             % becomes:  2822.9
%  and check that the analytic solution agrees
Pzc_an = ( U * sqrt(7)/5 )^2 / R   % also  2822.9
```

Warning: in the above program-code, the underscore (_) and caret symbol ('hat', ^) might not copy nicely from the pdf into Octave/Matlab; just write them instead. Or download the file again, but using `.tex` instead of `.pdf` extension to the name, and find this text in the LaTeX code.

b) How much power is dissipated in Z_c, if $Z_c = Z_b = R$ and $Z = -\mathrm{j}R$.

This choice of impedances implies that the balanced three-phase load consists of capacitors, while the two other impedances are still resistors. All five impedances have the same magnitude.

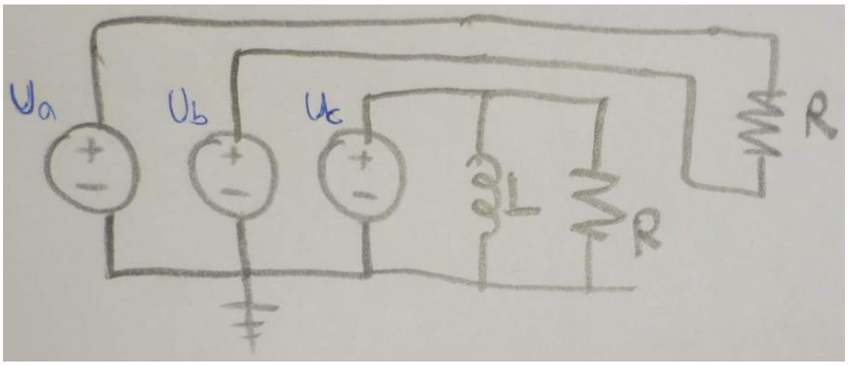
Using the computer-based method,

```
U = 400;  R = 15.87;   s = exp(-1j*2*pi/3);
Ua = U/sqrt(3);  Ub=s*Ua;  Uc=s*Ub;
Zc = R;   Zb = R;   Z = -1j*R;
vx = ( Uc/Zc + Ub/Zb ) / (1/Zc + 1/Zb + 3/Z)
Pzc = real( abs(Uc-vx)^2 / conj(Zc) )     % becomes:  1758.9
```

The ratio 1759/2823 between this case ($Z = -\mathrm{j}R$) and the initial case ($Z = R$) can be expected to be true for any choice of R, since the value of R affects all the KCL terms equally in size.

Exercise 3

Assume, as usual, that the sources produce a balanced 3-phase voltage, with phase-voltage magnitude U.



Find the complex power delivered by each of the three sources shown here.

Answer 3

Source a: Let's define the current out of the source's phase-a, towards the top of its connected resistor, to be i,

$$i = \frac{U_a - U_b}{R} = \frac{\sqrt{3}U\angle 30^\circ}{R}.$$

Then the power out of phase 'a' of the source is

$$S_a = U_a i^* = U\angle 0 \, \frac{\sqrt{3}U\angle{-30^\circ}}{R} = \frac{\sqrt{3}U^2\angle{-30^\circ}}{R} = \frac{\sqrt{3}U^2}{2R}\left(\sqrt{3} - \mathrm{j}\right).$$

Source b: Continuing with the same definition of i as above, the power out of phase 'b' of the source is

$$S_b = U_b(-i)^* = U\angle{-120^\circ} \, \frac{\sqrt{3}U^2\angle 150^\circ}{R} = \frac{\sqrt{3}U^2\angle 30^\circ}{R} = \frac{\sqrt{3}U^2}{2R}\left(\sqrt{3} + \mathrm{j}\right).$$

Source c: The two components connected to the source are L and R. These are directly parallel with the source, so they have the source's voltage across them.

The relation $S = |u|^2/Z^*$, for a voltage u applied to an impedance Z, tells us that this resistance absorbs a complex power of U^2/R (purely active), and the inductor absorbs a complex power of $\mathrm{j}\frac{U^2}{\omega L}$ (purely reactive).

These two components and source U_c have only one node that connects to other parts of the circuit, so they cannot transfer power to or from the rest of the circuit. The above values must therefore equal the total complex power supplied by source U_c:

$$S_c = U^2\left(\frac{1}{R} + \mathrm{j}\frac{1}{\omega L}\right).$$

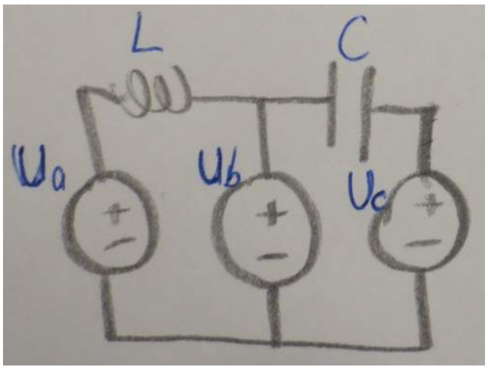
This could alternatively have been found by "KVL and KCL", by finding the phasor current from the source. As often happens, the power-based method was quicker by avoiding the need to handle angles.

A comment. It **may seem strange** that the sources a and b are together connected only to a resistor, and yet both have reactive power flows! Resistors cannot consume or generate reactive power (and reactive components cannot consume or generate active power).

Consider sources a and b, and the resistor between them. Together these form a loop that is independent of the other components in the circuit (it is connected to them only by one node). The current through the resistor is in phase with the voltage across the resistor — it has to be, by Ohm's law. So no reactive power is being produced or consumed in the resistor: its power-factor is 1. This same current passes through both of the sources in this loop. But each of these sources has a voltage that is *not* in phase with the total voltage of both sources together: if one compares U_a with $-U_b$, as it is $U_a - U_b$ (or $U_b - U_a$) that drives current in the loop, there is a 60° phase shift between these sources. This means that each source's voltage is out of phase with the current, in fact by 30°. So *both* sources have a reactive part of their power. Complex power is conserved in a circuit, so we expect that if one source delivers reactive power (source b here) then the other must absorb exactly this, since the remaining component R has zero reactive power. In contrast, in *balanced* conditions, all three phases of any source or load would be expected to have identical complex power, so any reactive power transfer must be between different three-phase sources or loads, not between phases of the same source or load.

Exercise 4

U_a, U_b and U_c are a three-phase source, giving line-voltage magnitude U (note! line-voltage) at angular frequency ω, and having conventional phase-rotation.



a) Find the complex power delivered *to* L and to C (both together).

b) Find the complex power delivered *by each* phase of the three-phase source, i.e. by each individual two-terminal source: a, b and c.

Hints/Ideas:

Try drawing a phasor diagram of the voltages and currents. You can choose any phase-reference, e.g. set the a-phase voltage to have zero angle. Notice that the impedances are across the line voltages, which have different phase angle from the phase voltages. The currents in these impedances are 90° shifted from their voltages, as they are purely reactive components.

If you prefer to solve numerically, then try that. In this way, you show you understand the concept, even if the complex algebra is still a trouble for you; but please try a bit of practice with rearranging the equations too. Set some values, e.g.,

```
U = 690, w = 2*pi*60, L = 0.18, C = 39e-6
```

in an environment such as Octave/Matlab, then set up phasors of voltage and thereby of current, from which to find complex power. This can also be used for checking algebraically derived equations.

Answer 4

a) Complex power delivered *to* L and to C.

Each of these components has a voltage of magnitude U across it, as it 'sees' the line voltage. Each therefore consumes a complex power of U^2/Z^* where Z is its impedance.

$$S_{\rm LC} = \frac{U^2}{-\mathrm{j}\omega L} + \frac{U^2}{\frac{1}{-\mathrm{j}\omega C}} = \mathrm{j}U^2\left(\frac{1}{\omega L} - \omega C\right).$$

A 'reasonableness check' can be made. We see the equation agrees with our expectation that purely reactive components can only deal with purely imaginary (reactive) power. If the capacitor becomes small, then the consumed reactive power becomes more strongly positive, which is correct for an inductive load (within the conventional but arbitrary definition that inductors 'consume' and capacitors 'produce' reactive power).

b) Complex power delivered *by* each phase of the three-phase source.

In each case, the voltage of the source, and the current out of the source's +-reference terminal can be used to find the complex power delivered by that source.

Let's start with phases 'a' and 'c', as only one component is connected to each.

PHASE-A: The current out of this source is

$$i_a = \frac{U_a - U_b}{\mathrm{j}\omega L} = -\mathrm{j}\frac{U_a - U_b}{\omega L},$$

and this results in a complex power output of

$$S_a = U_a i_a^* = U_a\left(-\mathrm{j}\frac{U_a - U_b}{\omega L}\right)^* = U_a \cdot \mathrm{j}\frac{U_a^* - U_b^*}{\omega L}.$$

(Notice here that the way to get a complex conjugate of an expression with lots of imaginary parts is to take the conjugate of every part, i.e. everything that was $\pm\mathrm{j}$ becomes $\mp\mathrm{j}$, and any polar form $\angle\pm\phi$ becomes $\angle\mp\phi$.)

Now, simplify and insert known quantities. We use the line voltage magnitude U, such that $|U_a| = |U_b| = |U_c| = U/\sqrt{3}$, allowing magnitudes of $|U_a U_b|$, $|U_a U_b^*|$ etc to be written $U^2/3$. The phase rotation is a,b,c, which means that $\angle U_a - \angle U_b = +\frac{2\pi}{3}$; hence, the angle of $U_a U_b^*$ is $\frac{2\pi}{3}$.

$$S_a = \mathrm{j}\frac{U_a {U_a}^*}{\omega L} - \mathrm{j}\frac{U_a U_b^*}{\omega L} = \mathrm{j}\frac{U^2}{3\omega L} - \mathrm{j}\frac{U^2}{3\omega L}\angle\tfrac{2\pi}{3} = \mathrm{j}\frac{U^2}{3\omega L}\left(1 - \mathrm{e}^{\mathrm{j}\frac{2\pi}{3}}\right) = \mathrm{j}\frac{U^2}{3\omega L}\left(1 - \left(\tfrac{-1}{2} + \mathrm{j}\tfrac{\sqrt{3}}{2}\right)\right).$$

The rectangular form of complex power delivered by the phase-a source is then

$$S_a = \frac{U^2}{3\omega L}\left(\tfrac{\sqrt{3}}{2} + \mathrm{j}\tfrac{3}{2}\right) = \frac{U^2}{2\omega L}\left(\tfrac{1}{\sqrt{3}} + \mathrm{j}\right).$$

This is a sort of opposite case to the earlier question where a resistor connected to a source carried imaginary power from the source. In our present question, the source connects into an inductor, but the complex power flowing from the source has an active as well as a reactive part. Neither the capacitor nor the inductor can produce or consume active power, so active power must be flowing into another source.

PHASE-C: The current out of this source is

$$i_c = (U_c - U_b)\ \mathrm{j}\omega C,$$

so its complex power output is

$$S_c = U_c i_c^* = U_c\ (\mathrm{j}\omega C\,(U_c - U_b))^* = U_c(-\mathrm{j}\omega C)\,(U_c^* - U_b^*) = -\mathrm{j}\omega C\,(U_c U_c^*) + \mathrm{j}\omega C\,(U_c U_b^*)\,,$$

which by similar simplifications and substitutions gives

$$S_c = -\mathrm{j}\frac{\omega C U^2}{3}\left(1 - \mathrm{e}^{-\mathrm{j}\frac{2\pi}{3}}\right) = \frac{\omega C U^2}{2}\ \left(\tfrac{1}{\sqrt{3}} - \mathrm{j}\right).$$

PHASE-B: This source has two components connected. One way to find its complex power output would be to find its current, and thus $S_b = U_b {i_b}^*$, as was done for the other sources.

Another way is to notice that this output must be equal to the sum of all consumption of complex power by the other components in the circuit:

$$S_b = S_{\mathrm{LC}} - S_a - S_c = \mathrm{j}U^2\left(\frac{1}{\omega L} - \omega C\right) - \frac{U^2}{2\omega L}\left(\tfrac{1}{\sqrt{3}} + \mathrm{j}\right) - \frac{\omega C U^2}{2}\ \left(\tfrac{1}{\sqrt{3}} - \mathrm{j}\right).$$

Simplifying,

$$S_b = -\frac{U^2}{2\sqrt{3}}\left(\frac{1}{\omega L} + \omega C\right)\ + \mathrm{j}\frac{U^2}{2}\left(\frac{1}{\omega L} - \omega C\right).$$

A numerical check can be made, to detect errors in combining and simplifying the equations. Other checks, for dimensions and reasonableness, should also be made! Notice **how few lines, and how little thinking** are needed in order to get the solution when using numeric values in a program that handles complex variables! Most of the code is about setting input variables and comparing results with the expressions we derived earlier.

```
U = 690;   w = 2*pi*60;   L = 0.18;   C = 39e-6;
s = exp(-1j*2*pi/3); Ua = U/sqrt(3);   Ub=s*Ua;   Uc=s*Ub;

% L+C
Slc = abs(Ua-Ub)^2/(-1j*w*L) + abs(Ub-Uc)^2*(-1j*w*C)   % directly from circuit
SlcEq = 1j*U^2*( 1/(w*L) - w*C )   % from the equation we derived

% src: a
ia = (Ua-Ub)/(1j*w*L)
Sa = Ua*conj(ia)
SaEq = (U^2/(2*w*L)) * ( 1/sqrt(3) + 1j )

% src: c
ic = (Uc-Ub)*1j*w*C
Sc = Uc*conj(ic)
ScEq = (w*C*U^2/2) * ( 1/sqrt(3) - 1j )

% src: b
ib = -(ia+ic)   % KCL
Sb = Ub*conj(ib)    % directly from circuit
SbA = Slc - Sa - Sc    % alternative method, by conservation of complex power
SbEq = -(U^2/(2*sqrt(3)))*(1/(w*L)+w*C) + 1j*(U^2/2)*(1/(w*L)-w*C )   % equation
```

Exercise 5

Here's an example based on a question in a forum [link] for electricians (elinstallatörer).
It involves an *unbalanced* load ... be careful.

Situation:
A three-phase three-wire system with line-voltage 208 V.[1]
Four identical single-phase resistive loads, each rated 2 kW at 208 V.
Therefore, three loads can form a balanced delta connection, but adding the fourth will cause one side of the delta to have more load than the other sides.

Question:
What current magnitudes flow in the three lines?
The answer to this question is important for choosing the cables and fuses (säkringar).
There will not be just one value of current for all the lines, because the load is unbalanced.
You can choose *which* two lines the fourth load is connected between; then say what current magnitude there is in each of the three lines.

Below, paraphrased, are the question and some of the answers from the forum.
Consider each suggested answer (A0–A4), and decide whether any of them is correct!

Q ... solve this problem i'm having in finding total current being used across the 3 phases ... 4 heater sets, each 208 volts 2000 watts, therefore 9.62 amps.

A0 {30, 30, 20} A I rounded up to 10 Amp ... assume 2 heater sets on AB, 1 on BC, and 1 AC ... I was wondering if it would trip the 30 A breaker because there was 30 amps on phases A and B, and 20 amps on Phase C; or am I wrong in thinking that there is 30 amps on A and B and 20 on C?

A1 {20, 10, 10} A You have 4 heaters, 10 amps each (rounded). If you connect 3 of the heaters AB, AC, BC you would have 10 amps on each phase. You can then add the 4 heater between any two lines, e.g. AB. Then AB will have an additional 10 amps. Load would be approx. AB 20 A, AC 10 A, BC 10 A.

A2 {27, 27, 18} A 27A and 18A respectively [this answer started with the false assumption that the extra heater is half the power of the balanced ones]

A3 {26.4, 26.4, 16.7} A Since 3 of the 2kw heaters will be on different phases, are purely resistive and balanced, could one treat them collectively as a single 3ph load?: 6000W / 208V / 1.732 = 16.7A, and add the 4th (9.6A/ph, say across AB phase) giving 26.4 A

A4 {25.4, 25.4, 16.7} A I did my own calcualtions and got 25.44A on two of the lines and a 16.65 on one of the three lines.

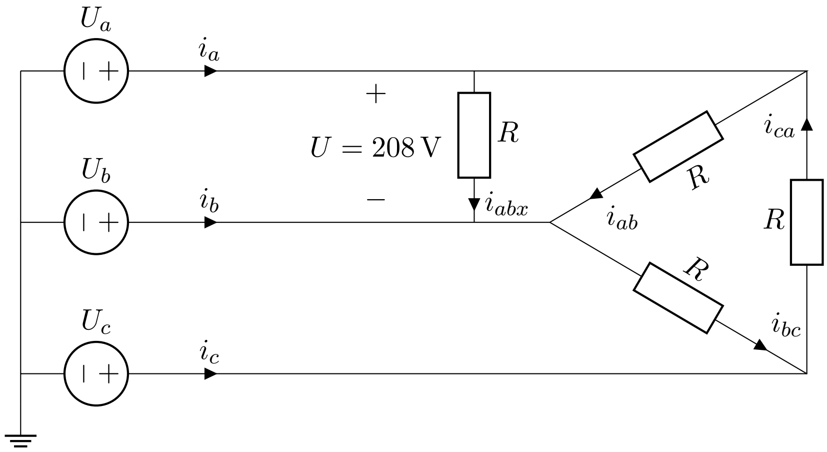
Notice how 'line' and 'phase' are often used interchangeably as names for the wires from the three-phase source. This is common in practical use, when people are interested in the network more than the load. Then phase-voltage often means the voltage from line to neutral, regardless of how sources or loads are connected to the line.

However, for consistency and clarity in this course, we've tried to use always 'line' for the wires, and 'phase' for the single-phase elements that three-phase sources and loads are made of.

[1] This is a common voltage in North America, as it provides a 120 V phase-voltage ($1/\sqrt{3}$) to a Y-connected load: it's normal there to have voltages around 120 V for lights and socket outlets (uttag). And yes, it will certainly be rms values that are meant in this forum, for all voltages and currents.

Answer 5

The last of the suggested answers (A4) was correct: 25.4 A in the lines that connect to the extra load, and 16.7 A in the line that doesn't connect to it. Thus, in the diagram below, $|i_a| = |i_b| = 25.4\,\text{A}$ and $|i_c| = 16.7\,\text{A}$. Each resistor is $R = \frac{U^2}{P} = \frac{(208\,\text{V})^2}{2\,\text{kW}} = 21.6\,\Omega$.



It's a slightly confusing calculation, at least if one has been used to doing only balanced or Y calculations. The difficulty is that the current in each line is the *phasor* sum (or difference, depending on how the directions are defined) of the currents in the loads that connect to that line. From earlier questions, we're familiar with doing that calculation for *balanced* Δ-loads: the line current is then $\sqrt{3}$ times the load's phase current. This factor comes from adding two phasors that differ in angle by 60°, which is the same as subtracting phasors that differ by 120° (i.e. 180° − 60°).

Thus, if just the three loads in balanced Δ are connected, the current magnitude in every line is $\frac{3 \cdot 2\,\text{kW}}{\sqrt{3} \cdot 208\,\text{V}} = 16.65\,\text{A}$. When the extra (single-phase) load is connected, as in the above diagram, then we can expect that line-c still has just $|i_c| = 16.65\,\text{A}$, as the extra load does not connect to that line.

The current magnitude $|i_{abx}|$ in the extra load is $\frac{2\,\text{kW}}{208\,\text{V}} = 9.62\,\text{A}$. We cannot just add this directly to 16.7 A to find the current in line a, because there is a 30° angle between them (draw a phasor diagram to confirm this!). We could make a diagram to help us add 9.62 and 16.65 as vectors with 30° difference; or we could use a computer: `abs( 16.65 + 9.62*exp(1j*pi/6) )`. The result is 25.4 A. The same current magnitude is found in line b, because there is a 30° angle between $-i_{abx}$ and the current to the balanced load in line b.

. . . continued . . .

Comments about the solution method:

We were not told that the *source* is Y-connected. But this is how we drew the diagram, above. It is often helpful for the thought-process to have this clear indication of potentials relative to a neutral; symmetries can be more obvious. If the system has just a three-wire connection to the source, then the only thing that affects the voltages and currents in the system is the relative voltages between those three wires. Our choice of a potential is for convenience in thinking.

Notice what happens in the equations (see the computer equations on the next page) if we define the left node as 1 MV instead of 0, or if we define one line-conductor (wire) as zero. Every term for finding a voltage or a current or a power involves *differences* between the potentials that we defined, so the arbitrary potential we've defined gets cancelled.

In short criticism of the answers from the forum:

A0 appears to see each line current as the scalar (not phasor) sum of all currents in load phases that are connected to that line.

A1 adds the extra load as a scalar (as did A0) but also assumes that the balanced part of the load has line currents equal to the load's phase currents, forgetting the $\sqrt{3}$ factor.

A2 is the same reasoning as A0, but using the wrong value for the extra load.

A3 correctly calculates line currents for a Δ balanced load, but then adds the extra single-phase load as a scalar; it's not terribly wrong, as 30° is quite small ... sort of.

A4 doesn't explain the reasoning, but gets it right ... clearly a seasoned handler of three-phase calculations.

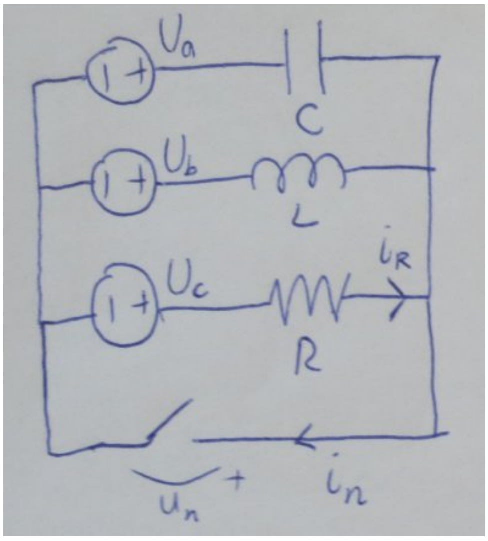
Here follows an example of the 'brainless way' (sensible!), where we don't bother with simplifications and trigonometry, but just work step by step, using complex numbers in a computer, and defining further variables as needed. This gives a good chance of checking that the intermediate steps are reasonable, and of easily adapting the code to a changed circuit.

```
% set up three-phase voltages
va = 208/sqrt(3);  % arbitrary choice of phase-a as angle-reference
vb = va*exp(-1j*2*pi/3);  % delayed (phase-lag) 120 degree
vc = vb*exp(-1j*2*pi/3);  % further delay
R = 208^2 / 2000;
% currents in the PHASES OF THE LOAD
iab = (va-vb) / R;
iabx = (va-vb) / R;  % in the extra load
ibc = (vb-vc) / R;
ica = (vc-va) / R;
% current magnitudes in the LINES
iA = abs(iab - ica + iabx)     % 25.4
iB = abs(ibc - iab - iabx)     % 25.4
iC = abs(ica - ibc)            % 16.7
```

Exercise 6

The sources in this circuit form a (balanced) three-phase source.
Each one has (rms) magnitude U.
Let us define $\underline{/U_a} = 0$, i.e. $U_a = U\underline{/0}$.

The phase-rotation is a,b,c, i.e. $U_b = U_a\,\mathrm{e}^{-2\pi/3}$ and $U_c = U_b\,\mathrm{e}^{-2\pi/3}$. (Notice that in this question, even the magnitude of the results depends on the phase-rotation, because the load has a different angle between current and voltage for each of its impedances.)



a) Find i_{R} with the switch closed.

b) Find i_{n} with the switch closed.

c) Find u_{n} with the switch open.

Answer 6

a) Find i_{R} with the switch closed.

With the switch closed it is easy, as each component is directly coupled to the voltage provided by just one source. Its voltage is therefore fixed, so its current is easily calculated. We weren't asked for just magnitudes, so we have to be careful to answer with a phasor, using the same reference angle as the question told us (phase-a source is defined as zero angle).

$$i_{\mathrm{R}} = \frac{U_c}{R} = \frac{U\underline{/-240^\circ}}{R} = \frac{U}{R}\underline{/-240^\circ}.$$

b) Find i_{n} with the switch closed.

Here again, the closed switch makes it quite simple to calculate, except for the manual complex arithmetic. Each impedance is in parallel with one source, so its current is determined. By KCL with these currents, the sought current i_{n} can be found:

$$i_{\mathrm{n}} \;=\; \frac{U_a}{\frac{1}{\mathrm{j}\omega C}} + \frac{U_b}{\mathrm{j}\omega L} + \frac{U_c}{R} \;=\; \mathrm{j}\omega C U + \frac{1}{\mathrm{j}\omega L}U\mathrm{e}^{-\mathrm{j}\frac{2\pi}{3}} + \frac{1}{R}U\mathrm{e}^{-\mathrm{j}\frac{4\pi}{3}}.$$

The angles (arguments) of the terms in this equation are determined by the combination of j and $\mathrm{e}^{\mathrm{j}\alpha}$ parts; these can be combined,

$$i_{\mathrm{n}} \;=\; U\left(\omega C\,\mathrm{e}^{\mathrm{j}\frac{\pi}{2}} + \frac{1}{\omega L}\,\mathrm{e}^{\mathrm{j}\left(\frac{-2\pi}{3}+\frac{-\pi}{2}\right)} + \frac{1}{R}\,\mathrm{e}^{-\mathrm{j}\frac{4\pi}{3}}\right),$$

and then the sines and cosines of these angles can be used to split the whole equation into real and imaginary parts,

$$i_{\mathrm{n}} \;=\; U\left(\frac{1}{R}\cos\tfrac{-4\pi}{3} + \frac{1}{\omega L}\cos\tfrac{-7\pi}{6} \;+\; \mathrm{j}\frac{1}{R}\sin\tfrac{-4\pi}{3} + \mathrm{j}\frac{1}{\omega L}\sin\tfrac{-7\pi}{6} + \mathrm{j}\omega C\right).$$

The angles are convenient ones (they are 30° away from the real or imaginary axes), whose sine and cosine give familiar values of $\frac{-1}{2}$ or $\frac{\sqrt{3}}{2}$, which can now be substituted,

$$i_{\mathrm{n}} \;=\; U\left(\frac{-1}{2R} + \frac{-\sqrt{3}}{2\omega L} \;+\; \mathrm{j}\left(\frac{\sqrt{3}}{2R} + \frac{1}{2\omega L} + \omega C\right)\right).$$

This seems about as simple as we can make it look. It's in rectangular form, and could easily be converted to polar form if so desired.

c) Find u_{n} with the switch open.

Now it gets nastier. The open switch is between the star-point of the source and the star-point of the three load-impedances. If the source and load were both balanced three-phase components, these two nodes would have equal potential even with the switch open, and so the marked voltage would be $u_{\mathrm{n}} = 0$.

But the load is definitely *not* balanced: each of the three impedances is a different type of component. So the voltage u_{n} is not zero.[2]

[2] Ok — we *could* choose an unbalanced source also, with just the right parameters so that the resulting u_{n} caused by the unbalanced source and load together will be zero. But an unbalanced load and balanced source are guaranteed to give $u_{\mathrm{n}} \neq 0$.

We can calculate u_n in the usual way, by nodal analysis based on one KCL equation. It isn't really necessary to define potentials, as we already have enough voltages marked to write the KCL using u_n as the unknown quantity:

$$\frac{U_a - u_\mathrm{n}}{\frac{1}{\mathrm{j}\omega C}} + \frac{U_b - u_\mathrm{n}}{\mathrm{j}\omega L} + \frac{U_c - u_\mathrm{n}}{R} = 0.$$

Rearranging,

$$\mathrm{j}\omega C U_a + \frac{U_b}{\mathrm{j}\omega L} + \frac{U_c}{R} = u_\mathrm{n}\left(\mathrm{j}\omega C + \frac{1}{\mathrm{j}\omega L} + \frac{1}{R}\right).$$

It's interesting to think what the terms in this equation represent! Consider the Thevenin (or Norton) equivalent seen at the two terminals of the open switch. The voltage we are trying to find is the open-circuit voltage of this two-terminal circuit, which is the ratio of short-circuit current to source-admittance. Its short-circuit current is the left-hand side of the above equation, which is the same as the current i_n from subquestion 'b'; that is not surprising, as the switch was then closed — a short-circuit! The term in parentheses on the right-hand side is the source admittance, i.e. $1/Z_\mathrm{T}$. This can be seen by the method of zeroing the independent sources (change the three voltage-sources to short-circuits), and calculating the equivalent impedance seen across the Thevenin terminals.

Using the simplifications that we already made in subquestion 'b', the previous expression can be written as

$$u_\mathrm{n} = U\,\frac{\frac{-1}{2R} + \frac{-\sqrt{3}}{2\omega L} + \mathrm{j}\left(\frac{\sqrt{3}}{2R} + \frac{1}{2\omega L} + \omega C\right)}{\frac{1}{R} + \mathrm{j}\left(\omega C - \frac{1}{\omega L}\right).}$$

We can make a quick check that the equations were rearranged correctly, by putting in some values of the 'known quantities', and comparing the results of each of our final expressions with the result from an earlier form of the expression that can more easily be seen to represent the circuit:

```
% set up some arbitrary values of variables
U = 100;  w = 10;
s = exp(-1j*2*pi/3);
Ua = U;  Ub=s*Ua;  Uc=s*Ub;
R = 10;  L = 0.9;  C = 0.011;

% part `a'
iR = Uc/R
iR_ = (U/R)*exp(-1j*4*pi/3)

% part `b'
in = U * ( -1/(2*R) + -sqrt(3)/(2*w*L)  ...
           + 1j*( sqrt(3)/(2*R) + 1/(2*w*L) + w*C ) )
in_ = Ua*1j*w*C + Ub/(1i*w*L) + Uc/R

% part `c'
un = U * ( -1/(2*R) + -sqrt(3)/(2*w*L) ...
          + 1j*( sqrt(3)/(2*R) + 1/(2*w*L) + w*C ) ...
         ) / ( 1/R + 1j*(w*C-1/(w*L)) )
un_ = ( 1j*w*C*Ua + Ub/(1j*w*L) + Uc/R ) / ( 1j*w*C + 1/(1j*w*L) + 1/R )
```

If we rearranged it correctly, then the differences such as `un - un_` will be near the level of the program's precision, i.e. 10^{-14} or less. And they are. So we probably did.

Exercise 7

Viktigt när du byter spis!
Elsäkerhetsverket (2015-11-20) warns `[artikel]`:

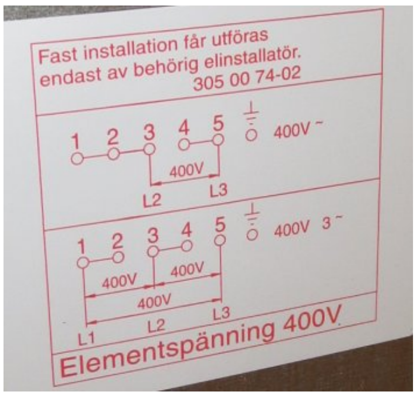
> Elsäkerhetsverket har uppmärksammat ett problem som kan uppstå då man byter ut en äldre spis som är ansluten med stickpropp i ett så kallat Perilex uttag, mot en ny.
>
> För vissa äldre spisar som var kopplade för huvudspänning 3 fas 400V, behövdes inte anläggningens neutralledare för spisens funktion. Det innebär att det förekommer installationer där neutralledaren saknas i spisuttaget.
>
> Nya spisar behöver en neutralledare i sitt uttag. Finns inte det så riskerar man att spisen förstörs. Spänningen kan bli upp till 400V istället för som avsetts 230V.

More about the Perilex plug/socket is on `[sv.wikipedia]`:

> Perilex är produktnamnet för en typ av fempoligt kontaktdon som är vanligt förekommande vid trefas-anslutningar till spisar och spishällar. Perilexkontakten finns i 16 A och 25 A utförande. Vanligtvis är kontaktdonet anslutet för 3-fas 230/400 volt med neutralledare och skyddsjord. Andra mer ovanliga användningsområden är uttag för likriktare till järnvägsbommar eller 400 Hz, 115/200 volt i flyghangarer.
>
> Då spisar utan elektronik vanligtvis inte använder neutralledaren var det förr vanligt att man inte heller drog fram en sådan till Perilex-uttaget. Då moderna spisar och spishällar med elektronik, så som induktionshällar, inte fungerar och i värsta fall förstörs om de kopplas in utan anslutning till neutralledaren, så är det viktigt att man kontrollerar sitt uttag innan man ansluter dessa.

A Perilex plug with just *three* pins (and a flat earth-pin). The socket on the wall actually has a neutral connected, but the cooker attached to this plug does not use the neutral.

A ‘traditional’ elspis (in English: electric cooker or stove or range, depending on the size and country!) in Sweden has heating elements that are resistors designed for 400 V. There may easily be 7 such elements: 4 on the hob (spishäll), and three for the top, bottom and radiant (‘S’) heat of the oven. Some hob elements have a smaller central ‘zone’ that can be used by itself for smaller pans, surrounded by a further zone that can be activated as well for larger pans. Ovens may have a lot of heating options: top and bottom, radiant, bottom only, etc; the switch usually makes it impossible to have all these installed elements active at the same time.



Each element is a single-phase load, that is connected between two line-conductors of the supply, in order to receive the system’s line voltage of 400 V. Usually the cooker is connected to all three phases, and has a few elements connected between each pair of phases. The result is that with all the elements on, the cooker will be an approximately-balanced Δ-connected three-phase resistive load. Alternatively, they can all be connected to a ‘single-phase’ 400 V supply, i.e. just two line conductors, as shown in the upper diagram in the photograph above. This will require a larger size of wire and fuse to carry the current when all elements are in use.

The following table shows the currents in the three line-conductors to a quite traditional cooker (Husqvarna regina), which is what was shown in the earlier pictures. It has plain resistive elements rated at 400 V. The line-conductors are marked L1, L2 and L3, which is normal terminology on equipment. A neutral conductor would be marked N; but this cooker is designed not to need a neutral.

case#	symbol	$I_{L1}/[A]$	$I_{L2}/[A]$	$I_{L3}/[A]$	$P/[kW]$	$R/[\Omega]$
1		2.9	0	2.9	1.2	140
2		5.2	0	5.2	2.1	80
3		2.5	0	2.3	0.9	170
4		4.6	0	4.6	1.8	90
5		0	2.7	2.7	1.1	150
6		0	5.7	5.8	2.3	70
7		0	3.1	3.1	1.2	130
8		1.2	1.2	0	0.7	230
9		4.6	4.7	0	1.8	87
10		4.9	4.9	0	1.9	83
11		5.0	5.0	0	2.0	81
12		7.0	7.0	0	2.8	58
13		8.0	8.0	0	3.2	50

Cases 1 – 7 show each possible choice on the hob: the hob has four parts, and three of these give the choice of heating just a smaller central zone, or the full size, to match the size of pan.

Cases 8 – 13 are the different options for the oven: only one can be chosen at a time. The oven's fan takes negligible current, around 0.1 A, between L1 and L2. All measurements were made with a clamp-on `current probe` connected to a multimeter; this combination gave 0.1 A as the smallest change visible on the display. The displayed values are shown in the table without modification.[3]

a) When the elements of cases 1 and 7 are both on, the current magnitude in L3 is 5.2 A. What current magnitude do you expect in L3 when the elements of cases 5 and 7 are both on?

b) And what current magnitude do you expect in L3 when the elements of cases 2 and 6 are both on?

c) What current magnitudes do you expect in each of { L1, L2, L3 } when as much load as possible is on, i.e. cases { 2, 4, 6, 7, 13 } all together.

[3] A comment about displayed values from the meter. In a few cases (e.g. case# 3) the values don't quite look like fitting KCL. All 13 cases were measured for L1, then L2, then L3. The heaters' resistances get rather higher when they're hotter, which can explain why later readings of current may be lower. The line voltage was around 405 V just after the current measurements, but was not checked before or between them; changes in supply voltage between these measurements might also have helped to push the reading up or down to the next number.

From the warning quoted at the start of this exercise, we infer that some modern cookers may have elements that are designed for 230 V, i.e. the line-neutral voltage ('phase voltage') of $\frac{1}{\sqrt{3}} \cdot 400\,\text{V}$.

That would allow the cooker to be used in a wider range of systems, including those where only 230 V is available. The cooker with all its elements turned on is then an approximately balanced Y-connected three-phase four-wire load.

(It may also, or instead, be that some elements are 400 V but the 230 V voltage from between line and neutral is used for running electronic controls and perhaps induction hobs if these are already designed for the commonly used 230 V level of normal wall-sockets.)

d) Suppose that the cooker described earlier is changed to a 230 V model, so that the switches and element powers are the same, but the elements are designed for giving their rated power at 230 V instead of 400 V.
What resistance would the element in case 13 (3.2 kW) need?
What current would it draw at 230 V?

As explained in the earlier quotations, one sometimes finds cooker sockets (spisuttag) that were installed with the assumption that the neutral was not needed.

If the neutral is not connected to a cooker that has Y-connected 230 V elements, this can cause a problem: an imbalance of the loading in the Y-connection will make the star-point change its potential, so some elements will see more than 230 V, and others less. Electronics for controllers or induction hobs seem likely to be even more easily damaged than simple resistive elements, by excessive voltage.

Consider again a 230 V model of the cooker, as described in subquestion 'd'. Assume that elements are grouped in the same way as in the 400 V model. For example, the ones connected L1-L2 (in the 400 V model) could be connected L1-N (in the 230 V model), then those that are L2-L3 could be L2-N, and those that are L3-L1 would then be L3-N. This 230 V model is unfortunately plugged into a socket that has no neutral connection.

e) In the above-described 230 V model with missing neutral, the two elements of cases 6 and 7 are turned on together. (All other elements are off. The cooker has no further loads such as controllers.) How much heat power does each of these two elements now produce? You can assume each element is a fixed resistance, regardless of the applied voltage.

f) Repeat 'e', but for cases 4 and 9.

g) Repeat 'e', but for cases 3 and 13.

Answer 7

a) Looking at the table, both these elements (cases 5 & 7) are between L2 and L3. Their combined current magnitude in L3 will therefore be the sum of their individual magnitudes, as their currents are in phase with each other. So $I_{\mathrm{L3}} = 2.7\,\mathrm{A} + 3.1\,\mathrm{A} = 5.8\,\mathrm{A}$. This expected value was confirmed by measurement. Notice that in a more general case we can't assume the total current magnitude is the scalar sum of individual magnitudes even if two loads are connected between the same two lines: we would have to check that the loads have the same power-factor. In this particular case, we assume all elements are resistors, so if they have the same voltage then their currents will be in phase with each other.

b) These elements (cases 2 & 6) are connected to L3-L1 and L3-L2 respectively. They therefore are exposed to voltages with a phase-difference. Assuming they are both resistors, their currents will have this same phase-difference, which must be considered when summing them: the sum's magnitude will be less than the sum of their magnitudes.
If we define the phase-voltages (potentials of the line conductors relative to neutral) as v_1, v_2, v_3, then the angles of the currents are the same as the angles of $v_3 - v_1$ and $v_3 - v_2$, which are 60° apart. Adding 5.2 A and 5.8 A (or 5.7 A) at 60° gives a result with magnitude $I_{\mathrm{L3}} = 9.5\,\mathrm{A}$, which is the value that was measured.

c) With the highest possible power consumption (cases { 2, 4, 6, 7, 13 } together), each of the line currents is the 60° phasor sum of the currents through the elements connected to each of the other lines.

```
IL1 = abs( 5.2+4.6 + exp(1j*pi/3)*(8.0) )
IL2 = abs( 5.7+3.1 + exp(1j*pi/3)*(8.0) )
IL3 = abs( 5.2+4.6 + exp(1j*pi/3)*(5.8+3.1) )
```

These calculated values, for { L1, L2, L3 } are { 15.4, 14.6, 16.2 } A .
The measured values were { 15.3, 14.5, 16.1 } A, possibly due to elements warming up.

That is quite good balance between the phases. If you don't see why the 60° angle occurs, try drawing the phasors of all three line potentials with respect to the neutral, i.e. the L-N voltages. Then draw the line voltages as phasors that go *between* these potentials: the phase-currents in the resistive loads will have the same angles as these line-voltages. Any pair of these has a 60° angle.

d) About $17\,\Omega$ and 14 A. We expect the resistance to be decreased by a factor 3 to maintain the same power if the voltage decreases by $\sqrt{3}$, because of the relation $P = u^2/R$. Alternatively we could think of Δ-Y conversion, in which the factor 3 occurs for the same reason.

e) Nothing happens in either element, if just 6 and 7 are turned on in this 230 V model of the cooker with a lost neutral. They are in parallel, but the terminal connected to the cooker's neutral point has no way for current to flow; there is no neutral conductor, and no elements connecting to other lines are turned on. The neutral point would then be at the same potential as the line that supplies these elements (L2 in the example given). So, zero power in each.

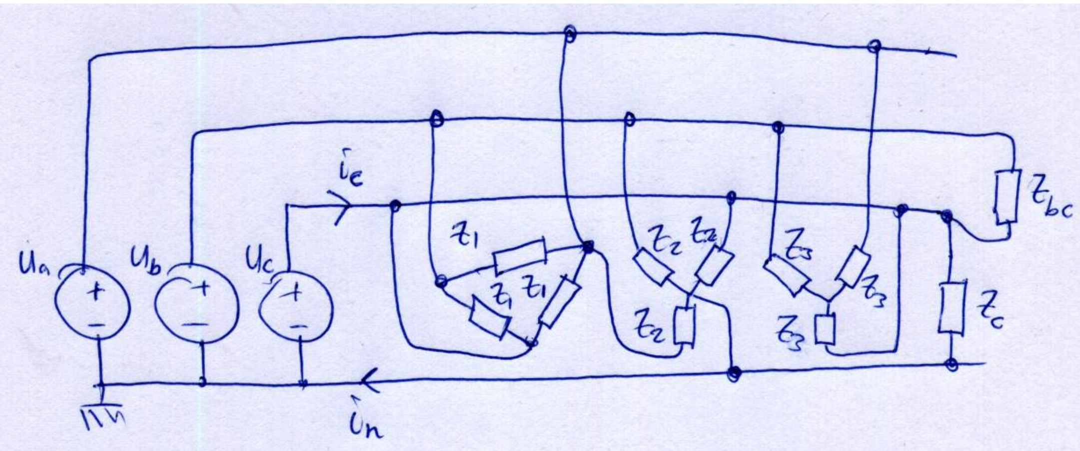
f) These elements (4 & 9) have very similar resistance (and therefore similar power rating), so they will share the voltage approximately evenly. But the voltage across the pair is just 400 V, not $2 \times 230\,\mathrm{V}$; each element gets 200V, meaning that its power is $\left(\frac{200\,\mathrm{V}}{230\,\mathrm{V}}\right)^2$ as much as the rated value of 1.8 kW. This is about 1.4 kW each.

g) In this case (3 & 13) the result is again that two elements become series connected between two lines, so they divide 400 V between them. But this time they're far from being evenly matched! By voltage division, the lower-power (higher-resistance) element has 310 V and the other has 90 V. Their powers will be 180% and 15% of the rated values, respectively, based on $\left(\frac{310\,\mathrm{V}}{230\,\mathrm{V}}\right)^2$ and $\left(\frac{90\,\mathrm{V}}{230\,\mathrm{V}}\right)^2$. The answer is 1.6 kW for #3, and 0.5 kW for #13 .

Exercise 8 Different load-types, some unbalanced load. *From EI1120 HW4 2013.*

The crudely sketched diagram below shows a 3-phase 4-wire supply system modelled as three ideal sources of sinusoidal voltage at angular frequency ω. These give phase-voltages that can be represented as phasors such that $u_\mathrm{a} = U\underline{/0}$, $u_\mathrm{b} = U\underline{/-2\pi/3}$, and $u_\mathrm{c} = U\underline{/-4\pi/3}$, where U is the rms (effective) value of the sinusoidal voltage.

Note that there are three *balanced* 3-phase loads: load 1 has impedances Z_1 in Δ (delta) connection; load 2 has impedances Z_2 in Y ('wye', or 'star') connection with the centre ('star-point') connected to the supply's neutral conductor; load 3 has impedances Z_3 star-connected with no neutral connection.



The only other two components shown are a single impedance Z_c from phase 'c' to neutral (the earthed conductor at the bottom), and another impedance Z_bc between phases b and c.

a) For each balanced load (1,2,3) write an expression for the load's total (all three impedances) complex power. Then write expressions for the complex power into the other loads Z_c and Z_bc.

b) What is the current i_c?
Express it as a phasor, in terms of the impedances and the rms phase-voltage magnitude U.

c) What is the current i_n?

d) [Very simple reminder of rms and 3-phase basics.]
If the rms phase-voltage magnitude of the source is U, what are the following magnitudes:

i) the peak value of the phase-voltage?

ii) the rms line-voltage (voltage between two of the line conductors such as the top and middle)?

iii) what is the peak of the line voltage?

Answer 8

The 3-phase 4-wire supply system in this question gives phase-voltages that can be represented as phasors such that $u_\mathrm{a} = U\underline{/0}$, $u_\mathrm{b} = U\underline{/-2\pi/3}$, and $u_\mathrm{c} = U\underline{/-4\pi/3}$, where U is the rms (effective) value of the sinusoidal voltage.[4]

There are three *balanced* 3-phase loads: load 1 has impedances Z_1 in Δ (delta) connection; load 2 has impedances Z_2 in Y ('wye', or 'star') connection with the centre ('star-point') connected to the supply's neutral conductor; load 3 has impedances Z_3 3 star-connected with no neutral connection.

The only other two components shown are a single impedance Z_c from phase 'c' to neutral (the grounded conductor at the bottom), and another impedance Z_bc between phases b and c.

Reminder: the complex power into an impedance Z can be calculated from the applied voltage by

$$ui^* = u\left(\frac{u}{Z}\right)^* = \frac{uu^*}{Z^*} = \frac{|u|^2}{Z^*},$$

where u and i are the rms phasor voltage across and current through the impedance, using the passive convention. Note that the result is dependent only on the absolute value (not the phase) of the voltage.

a) Calculate complex powers into the loads.

Loads 1,2,3 (balanced loads).

Load 1: Δ-connected: Each impedance Z_1 gets voltage $\sqrt{3}U$, and therefore has complex power $S = \left(\sqrt{3}U\right)^2/Z_1^*$. There are three such loads, so the total complex power is $S_\mathrm{t} = 9U^2/Z_1^*$.

Load 2: Y-connected: Each impedance Z_2 gets voltage U, thus complex power $S = U^2/Z_2^*$. The total complex power is thus $S_\mathrm{t} = 3U^2/Z_2^*$.

Load 3: Y-connected but '4-wire' (with neutral). This is exactly the same principle as load 2, because the load is balanced, and therefore there is no current in the neutral conductor.[5] Thus the total complex power is $S_\mathrm{t} = 3U^2/Z_3^*$.

We would have had nicer numbers if we'd used line voltage instead.[6] In general, for balanced

[4]Why have a phasor whose amplitude is an rms value instead of a peak value? This might sound wrong. We first introduced the idea of phasors by thinking of representing a signal like $A\cos(\omega t + \phi)$ as a complex quantity $A\mathrm{e}^{\mathrm{j}\omega t}\mathrm{e}^{\mathrm{j}\phi}$ and then removing the term $\mathrm{e}^{\mathrm{j}\omega t}$ which is assumed to be the same for all currents and voltages in the circuit (in steady-state excitation by independent sources with frequency ω). In that case, A was the amplitude, but now we're talking about making the phasor's size represent the rms value – how can we do this? The reasoning is similar to why we can drop the $\mathrm{e}^{\mathrm{j}\omega t}$: if we divide all the currents and voltages in ac analysis by the *same* function, we don't change the solution. By removing the $\mathrm{e}^{\mathrm{j}\omega t}$ we got the advantage of simpler equations. By dividing all currents and voltages by $\sqrt{2}$ we still get the same equations to solve for the circuit, but we get the advantage that when we multiply a voltage and [conjugate]current to get power, the result doesn't have to be scaled by a factor of two: instead it immediately tells us the power. That is why people almost always use rms values for phasors, within the subject of electric power.

[5]If the loads were not balanced then the 3-wire case would be harder but the 4-wire case would be fairly easy as the voltages across all impedances would be known directly. In the unbalanced Y-connected 3-wire case, one does not immediately know the potential on the 'star-point' (centre-connection), and therefore has to use Y-Δ conversion or e.g. node-analysis to get a result.

[6]If U_L is the line-voltage (rms absolute value), then the power in a Δ-connected load of 3 identical impedances Z_Δ would be just $S_\mathrm{t} = 3U_\mathrm{L}^2/Z_\Delta^*$, and in the Y-connected load would be $S_\mathrm{t} = 3\left(U_\mathrm{L}/\sqrt{3}\right)^2/Z_\mathrm{Y}^* = U_\mathrm{L}^2/Z_\mathrm{Y}^*$.

three-phase load, if we know the line voltage U_L and the current I in each phase-conductor of the line, then the total apparent power is $\sqrt{3}U_\mathrm{L}I$.

Unbalanced loads Z_c and Z_bc:

Load c: $S = U^2/Z_\mathrm{c}^*$.

Load bc: $S = \left(\sqrt{3}U\right)^2/Z_\mathrm{bc}^*$.

b) The current i_c, expressed as a phasor.

If we write the full equations, in the way we would normally have done before we specifically studied 3-phase circuits, then we would sum together contributions due to 8 impedances that affect the current in the c-phase conductor. The only impedances that don't matter are the top Z_1 in the delta (because it doesn't couple to phase c) and the two top Z_2 in the 4-wire Y (they also are not affecting phase c). All the others would be included in the calculation, although we know that several of them will in fact cancel because of being balanced.

However, we now know clever tricks about 3-phase systems, useful in the *balanced* state. For the three balanced loads we can simply take the total complex power, then assume $\frac{1}{3}$ of this in each phase, and thereby calculate the phase currents by dividing this complex power by the phase voltage.

For the three balanced loads this gives a contribution to the phase-c current of

$$\left(\frac{\frac{1}{3}\left(\frac{9U^2}{Z_1^*}+\frac{3U^2}{Z_2^*}+\frac{3U^2}{Z_3^*}\right)}{U\underline{/-4\pi/3}}\right)^* = \left(\frac{3}{Z_1}+\frac{1}{Z_2}+\frac{1}{Z_3}\right)\cdot U\underline{/-4\pi/3},$$

(the negative phase in the expression on the right comes from dividing by a negative phase [which gives positive phase] but then taking the complex conjugate [making it negative again!]...).

To this current we add the currents due to the two unbalanced loads, each of which is connected to phase c: these are

$$i_{\mathrm{z_{bc}}} + i_{\mathrm{z_c}} = \frac{u_\mathrm{c}-u_\mathrm{b}}{Z_\mathrm{bc}} + \frac{u_\mathrm{c}}{Z_\mathrm{c}} = \frac{1}{Z_\mathrm{bc}}\sqrt{3}U\underline{/\pi/2} + \frac{1}{Z_\mathrm{c}}U\underline{/-4\pi/3},$$

so the total is

$$i_\mathrm{c} = U\left[\left(\frac{3}{Z_1}+\frac{1}{Z_2}+\frac{1}{Z_3}\right)1\underline{/-4\pi/3} + \frac{\sqrt{3}}{Z_\mathrm{bc}}1\underline{/\pi/2} + \frac{1}{Z_\mathrm{c}}1\underline{/-4\pi/3}\right].$$

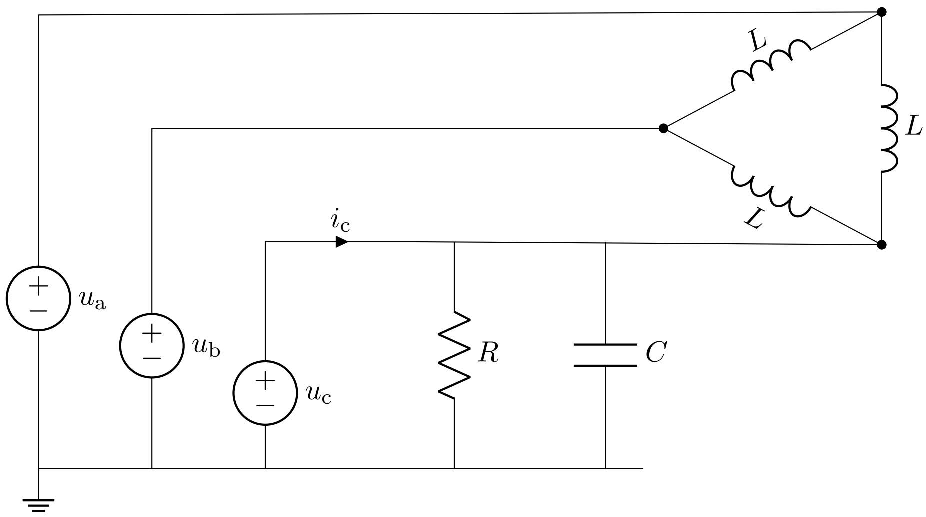
c) The current i_n, as a phasor (i.e. caring about the size *and* phase). We could write this out in full detail, by adding the terms from the 4 impedances that connect to the neutral conductor (3 of Z_2 and 1 Z_c. But this isn't necessary: the load 2 is balanced, so its neutral current is zero as long as the voltage source is a balanced three-phase source (equal magnitudes, perfect 120° phase-shifts). Thus, the only neutral current is the current in Z_c. This is $u_\mathrm{c}/Z_\mathrm{c}$. As we don't know the angle of the impedance, it's fine to write this as just $\frac{1}{Z_\mathrm{c}}U\underline{/-4\pi/3}$. We could alternatively write $i_\mathrm{n} = \frac{U}{|Z_\mathrm{c}|}\underline{/-4\pi/3 - \underline{/Z_\mathrm{c}}}$. In reality we tend to care about the magnitude of a current or voltage, but not about its phase-angle unless we need to add different currents or voltages together (in which case the phases are obviously very important).

d) Given that the rms *phase-voltage* (from a phase to neutral [ground]) from the source is U, the absolute values of other voltage-definitions are:
i) peak phase-voltage: $\sqrt{2}U$ (because it's sinusoidal).
ii) rms line voltage (voltage between two phases): $\sqrt{3}U$.
iii) peak line voltage: $\sqrt{2}\sqrt{3}U$.

Note, again, that the *normal* power-system way of describing a three-phase system is to quote the line-voltage. However, for some of our calculations it is convenient to use the phase-voltage (voltage from a line conductor to the neutral), because this gives us nicely defined potentials with respect to our 'reference node'. Above the lowest voltage-levels European practice is not even to have a neutral conductor, but to have all loads connected between the phases (any neutral or 'ground' connection becomes important only in faults where there is a short-circuit to the ground). However, North American practice is to include neutrals even at the 13.8 kV level that feeds mainly single-phase transformers down to 120 V. And in some sparsely populated areas such as Australia, there may be just a single wire in a line at the 11 kV level, with the current returning via electrodes in the ground. Lots of differences!

Exercise 9

An unbalanced system.



a) What choice of C will minimise the magnitude of the current i_c?

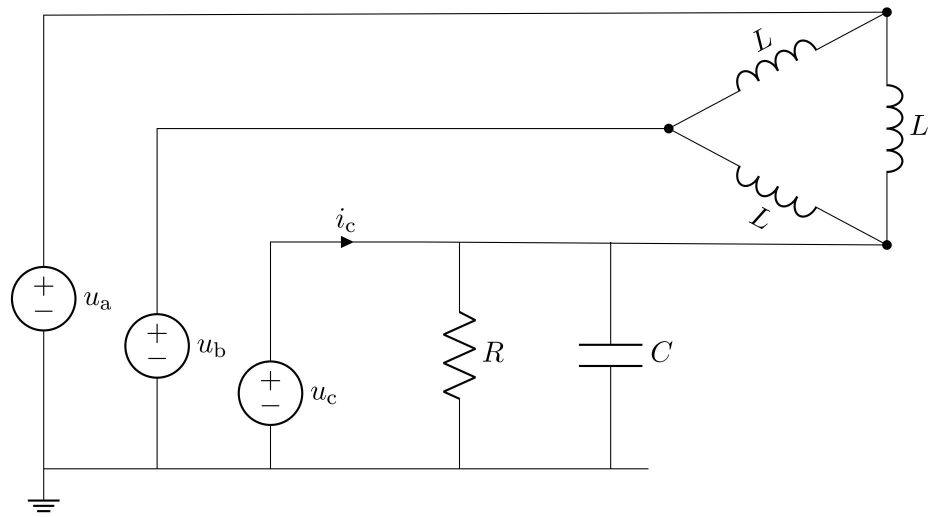
b) Assume C now has the value calculated in part 'a'.
Use that, along with the following numeric values, to calculate the *phasor* (magnitude *and* phase) of i_c.
You can do this numerically, i.e. by computer, to avoid having to do lots of sines and cosines by hand. But try it by hand as well, if you want ....

```
f = 50      % 50 Hz
w = 2*pi*f      % 314 rad/s
Up = 230      % rms phase-voltage of supply
ua = Up*exp(1j*0)
ub = Up*exp(-1j*2*pi/3)
uc = Up*exp(-1j*2*pi*2/3)

R = 16      % 16 ohm
L = 509e-3      % 509 millihenry
```

c) What would you expect about the relation between the phase-angle of u_c and i_c, in view of the condition required in part 'a'? Is this true in your result?

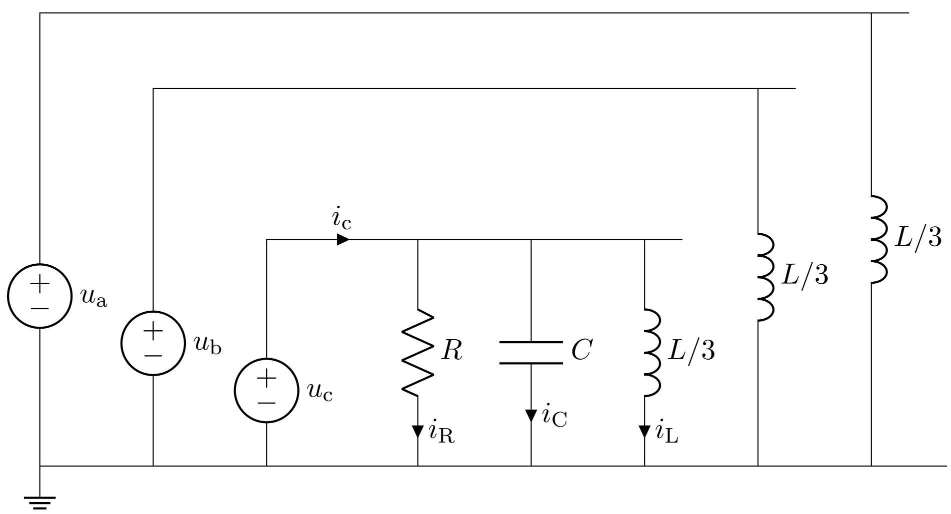
Answer 9

Solution of the unbalanced 3ph case.



a) What choice of C will minimise the magnitude of the current i_c?

In our case of an ideal 3-phase source and equal inductances, we know we can replace the Δ-connected inductors L with Y-connected ones of $L/3$, and can link their star-point to the neutral without changing i_c.



In the new circuit, we see i_c is the combination of three currents, which are all 90° shifted in phase from each other,

$$i_c = i_R + i_C + i_L = u_c \left(\frac{1}{R} + j\omega C - j\frac{1}{\omega L/3} \right)$$

Therefore,

$$|i_c| = U_p \sqrt{\left(\frac{1}{R}\right)^2 + \left(\omega C - \frac{3}{\omega L}\right)^2}.$$

The value of C to minimise $|i_c|$ is thus

$$\omega C = \frac{3}{\omega L} \qquad \Longrightarrow \qquad C = \frac{3}{\omega^2 L}.$$

An alternative method, instead of converting the inductors from Δ to Y, would be to do a phasor addition of the currents in the two inductors connected to the phase-c in the original circuit.

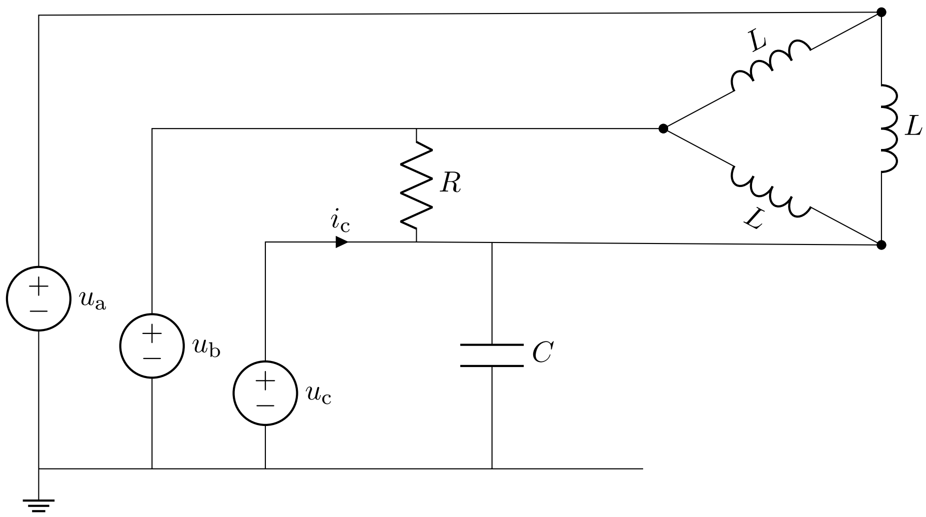
b) Assume C now has the value calculated in part 'a'.
Use that, along with the following numeric values, to calculate the *phasor* (magnitude *and* phase) of i_c.

```
% given values:
f = 50     % 50 Hz
w = 2*pi*f     % 314 rad/s
Up = 230     % rms phase-voltage of supply
ua = Up*exp(1j*0)
ub = Up*exp(-1j*2*pi/3)
uc = Up*exp(-1j*2*pi*2/3)
R = 16    % 16 ohm
L = 509e-3   % 509 millihenry

% analytic calculation of C
C = 3/(w^2*L)
   5.972e-5   % i.e. 60 uF

% phasor of current ic
ic = uc*( 1/R + 1j*w*C - 1j/(w*L/3) )
abs(ic)
  14.375
angle(ic)*180/pi
  120.00
```

c) We are choosing one reactive component, to minimise total current with a parallel connected resistor and opposite type of reactive component. It is clear that the reactive current will always add to the resistive current to make a bigger total current, as the reactive and resistive parts are always perpendicular (90°), and the hypotenuse of a right-angled triangle is bigger than either of the other sides. The lowest total current is therefore when we avoid all reactive current by cancelling it between the inductor and capacitor. So this question is actually a sort of power-factor correction question. When the power-factor correction is perfect, we can expect the phase-angle of u_c and i_c to be the same; the load should look like a plain resistor due to the cancelled currents between the reactive components. In this question, phase-c is defined as having a voltage at 120° (which is the same as −240°. This is seen to be the same phase-angle as the current calculated in part 'b', above; so this matches with what was expected in the above explanation.

Exercise 10

A harder variation on the previous question.



a) What choice of C will minimise the magnitude of the current i_c?

b) Assume C now has the value calculated in part 'a'.
Use that, along with the following numeric values, to calculate the *phasor* (magnitude *and* phase) of i_c.
You can do this numerically, i.e. by computer, to avoid having to do lots of sines and cosines by hand. But try it by hand as well, if you want ....

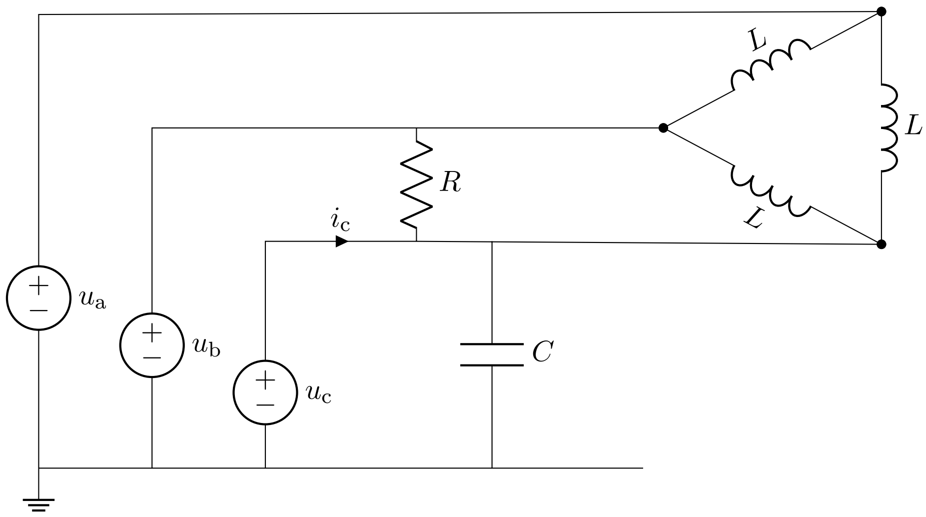
```
f = 50      % 50 Hz
w = 2*pi*f      % 314 rad/s
Up = 230      % rms phase-voltage of supply
ua = Up*exp(1j*0)
ub = Up*exp(-1j*2*pi/3)
uc = Up*exp(-1j*2*pi*2/3)

R = 16      % 16 ohm
L = 509e-3      % 509 millihenry
```

c) What would you expect about the relation between the phase-angle of u_c and i_c, in view of the condition required in part 'a'? Is this true in your result?

d) Now assume that C has instead been chosen so that $\frac{1}{\omega C} = \omega L$ (that should not have been your solution in 'a'!). What is the total complex power from the three-phase source? What is the phasor of current i_c? You **do not need** to do this symbolically with all the angles and phasor addition. Just do it to get a numeric result by continuing the Octave/Matlab code above.

Answer 10

Solution of the harder variation.



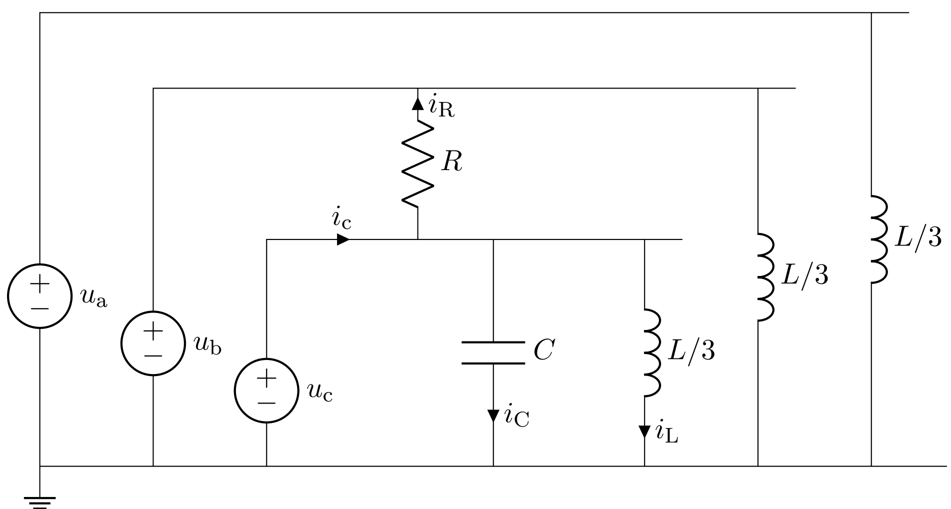
a) What choice of C will minimise the magnitude of the current i_c?

The current i_c comes out of the source and into four different paths: an R and L in parallel go to to phase b; another L goes to phase a; and C goes to ground (neutral).

In the most general case, where the sources a and b could have arbitrary magnitude and phase, and the inductances could be different, this question gets complicated. It might even turn out that i_c can be reduced to zero (see note at end of this document).

However, in our case of an ideal 3-phase source and equal inductances, we know we can replace the Δ-connected inductors L with Y-connected ones of $L/3$, and can link their star-point to the neutral without changing how the circuit behaves.

In this new circuit, i_c should be exactly as before, but there are just three paths for it to take, and two of them are parallel.



This is still actually rather tricky. The direct way to approach it is through current and voltage phasors. Let's take each of the three parts of i_c:

$$i_\mathrm{c} \quad = \quad i_\mathrm{R} + i_\mathrm{C} + i_\mathrm{L} \quad = \quad \frac{u_\mathrm{c} - u_\mathrm{b}}{R} + \mathrm{j}\omega C u_\mathrm{c} - \mathrm{j}\frac{3u_\mathrm{c}}{\omega L}$$

The voltage $u_\mathrm{c} - u_\mathrm{b} = u_\mathrm{cb}$ is a "line voltage". Its magnitude is $\sqrt{3}U_\mathrm{p}$, which is the standard relation between line-line and line-neutral voltages. Its phase is 90°. This can be seen by drawing the phasors u_c (at 120°, which is the same as −240°) and u_b (at −120°), and drawing the voltage u_cb (the line between their ends). The resistive current is then also at a phase of 90° as there is no phase-shift between voltage and current in a resistor.

The capacitor and inductor are both connected to the same voltage, u_c. This has a phase of 120°, from the definition of the voltage source. The current in the capacitor is therefore at a phase of $120° + 90° = 210° = -150°$. The current in the inductor is at a phase of $120° - 90° = 30°$.

Putting together all the above details of angles and magnitudes, the expression for i_c can be written as

$$i_\mathrm{c} \;=\; \frac{\sqrt{3}U_\mathrm{p}}{R}(\cos(90°) + \mathrm{j}\sin(90°)) + U_\mathrm{p}\omega C(\cos(210°) + \mathrm{j}\sin(210°)) + \frac{3U_\mathrm{p}}{\omega L}(\cos(30°) + \mathrm{j}\sin(30°))$$

and taking advantage (as usual) of the nice properties of sines and cosines of multiples of 30°,

$$i_\mathrm{c} \;=\; \frac{\sqrt{3}U_\mathrm{p}}{R}(\mathrm{j}) + U_\mathrm{p}\omega C\left(\frac{-\sqrt{3}}{2} + \frac{-\mathrm{j}}{2}\right) + \frac{3U_\mathrm{p}}{\omega L}\left(\frac{\sqrt{3}}{2} + \frac{-\mathrm{j}}{2}\right)$$

so

$$i_\mathrm{c} \;=\; \left(\frac{3U_\mathrm{p}}{\omega L}\cdot\frac{\sqrt{3}}{2} + U_\mathrm{p}\omega C\frac{-\sqrt{3}}{2}\right) + \mathrm{j}\left(\frac{\sqrt{3}U_\mathrm{p}}{R} + U_\mathrm{p}\omega C\frac{-1}{2} + \frac{3U_\mathrm{p}}{\omega L}\cdot\frac{1}{2}\right)$$

$$i_\mathrm{c} \;=\; \frac{U_\mathrm{p}}{2}\left[\sqrt{3}\left(\frac{3}{\omega L} - \omega C\right) + \mathrm{j}\left(\frac{2\sqrt{3}}{R} + \frac{3}{\omega L} - \omega C\right)\right]$$

We could waste lots of time playing with this to find an exact expression for a minimum. Square the real and imaginary parts (but we don't have to take the square root: we want to find where the minimum is, not the actual value $|i_\mathrm{c}|$ at the minimum), and add them. Differentiate, and find where the derivative is zero. Check it's a minimum (second derivative positive). Simplify and write out.

$$C = \frac{3R + \frac{\sqrt{3}}{2}\omega L}{\omega^2 LR}$$

That's a lot of work to do manually, and a high chance of making an error. More sensible in a practical case is just to find the minimum numerically by plotting the function $|i_\mathrm{c}| = f(C)$ for your given values of the other variables (L, ω, R): see the example in the answer to part 'b'.

The analytic solution has its beauty when we don't know the other variables, and want to know their effect. Even if we *don't* know some of the other variables, it's good to make a plot based on numerical solution with some arbitrary values of these variables, then compare this with the result from the analytic equation for the minimum, in order to verify that we didn't make a mistake in finding the analytic equation.

b) Assume C now has the value calculated in part 'a'.
Use that, along with the following numeric values, to calculate the *phasor* (magnitude *and* phase) of i_c.
You can do this numerically, i.e. by computer, to avoid having to do lots of sines and cosines by hand. But try it by hand as well, if you want ....

```
% Given values:
f = 50      % 50 Hz
w = 2*pi*f      % 314 rad/s
Up = 230     % rms phase-voltage of supply
ua = Up*exp(1j*0)
ub = Up*exp(-1j*2*pi/3)
uc = Up*exp(-1j*2*pi*2/3)
R = 16      % 16 ohm
L = 509e-3      % 509 millihenry

% Analytic calculation of C
C = (3*R + (sqrt(3)/2)*w*L) / (w^2*L*R)
% this gives  232e-6    (232 uF)

ic = (uc-ub)/R + 1j*w*C*uc - 1j*3*uc/(w*L);
% or could have used the rearranged expression,
%ic = (Up/2)*( sqrt(3)*( 3/(w*L) - w*C ) + 1j*( 2*sqrt(3)/R + 3/(w*L) - w*C ) )

abs(ic),
  21.56
angle(ic)*180/pi
  120.00

% Alternative much quicker numeric approach for one-variable optimisation
c = logspace(-6,1,10000);  % range of likely values
% make vector of ic at each of these values
% note: using the simple early equation, to avoid possible errors
% in the later manipulations: this makes it an even more useful check
ic = (uc-ub)/R + 1j*w*C*uc - 1j*3*uc/(w*L);
% find c at minimum [shorter but not so reliable would be c(abs(ic)==min(abs(ic))) ]
c( abs(abs(ic) - min(abs(ic))) < 10*eps )
%     2.3354e-04    % very close to analytic; improve by using more points
loglog( c, abs(ic) );  % if we want to check the function
```

c) Expectations about the relation between the phase-angle of u_c and i_c?

See the answer to the previous question, regarding that simpler case. Our situation, above, is not so obvious. The resistor connects to *another* phase so its current is in phase with the voltage u_{cb}, not u_c. However, we still found (part 'b') a current i_c at 120°, the same as $\angle u_c$. Very well done if you can explain this simply ... I am not going to try here.

d) Now assume that C has instead been chosen so that $\frac{1}{\omega C} = \omega L$. What is the total complex power from the three-phase source? What is the phasor of current i_c? You **do not need** to do this symbolically with all the angles and phasor addition. Just do it to get a numeric result by continuing the Octave/Matlab code above.

```
% Given values:
f = 50      % 50 Hz
w = 2*pi*f      % 314 rad/s
Up = 230      % rms phase-voltage of supply
ua = Up*exp(1j*0)
ub = Up*exp(-1j*2*pi/3)
uc = Up*exp(-1j*2*pi*2/3)
R = 16      % 16 ohm
L = 509e-3      % 509 millihenry
C = 1/(w^2*L)   % 20 microfarad

% Current as phasor (same equation as in part `b')
ic = (uc-ub)/R + 1j*w*C*uc - 1j*3*uc/(w*L);
abs(ic)
    26.454
angle(ic)*180/pi
    84.596

% Complex power.  Just sum the u^2/conj(Z) for every component!
S = (sqrt(3)*Up)^2 * ( 1/R + 3/(-1j*w*L) )  +  Up^2 * (-1j*w*C)
     9.9188e+03 + 2.6465e+03i
% i.e. about 10 kW, 2.65 kVAr

% Check: calculate the currents, and calculate complex power from
% all sources together:
ia = ua/(1j*w*L/3)
ib = ub/(1j*w*L/3) + (ub-uc)/R
S_ = ua*conj(ia) + ub*conj(ib) + uc*conj(ic)
     9.9188e+03 + 2.6465e+03i
```

Exercise 11

You are in a house supplied from a 400 V 50 Hz transformer, through a four-wire three-phase line that is 1 km long. Each conductor of the line is approximately a series resistance of 1 Ω and a series inductance of 1 mH. These values are typical for 'open wire' lines of bare copper conductors with 16 mm^2 area.

You are sitting by an incandescent type (glödlampa) lamp that is connected between phase-a and the neutral; it is rated at 230 V 40 W. Several activities are causing the lamp to change in brightness. Tolerating this annoyance temporarily, you start calculating how much the other loads are changing the voltage that the lamp receives. (Then you will fetch a voltmeter to check your calculation, and fetch a modern LED lamp that isn't so sensitive to changes in voltage.)

For the following loads, a–e, calculate how much the load changes the voltage *magnitude* at the lamp.
The transformer can be assumed to be an ideal three-phase voltage source, as the line's impedance is several times higher than the "source impedance" at the output of the transformer.[7]
When a load is specified in terms of power, this is assumed to be a power consumed at the intended (rated) voltage.

a) A 2 kW fan-heater also connected between phase-a and neutral keeps turning on and off every few minutes, controlled by its thermostat. How much change does this make to the lamp's voltage?

b) A 3 kW kettle (vattenkokare) is being used repeatedly by other people. It is connected from phase-b to the neutral. How much change does this make to the lamp's voltage?

c) An oven has a 2.5 kW heater that is connected between phase-b and phase-c. How much change does this make to the lamp's voltage?

d) Another 2.5 kW heating element on top of the cooker (spis) is connected between phase-a and phase-b.

e) A 1 kW three-phase induction motor (with no neutral wire) starts and stops automatically to keep a tank filled. During starting, such motors draw a much higher current, and at a lower power-factor, than when running at normal speed. This one has a power factor of 0.95 in normal operation. When starting, it take 8 times its normal current, and has a power factor of 0.5 (lagging). How much change does this make to the lamp's voltage?

[7] The source impedance seen at the transformer's LV (low-voltage, 400 V) output terminals is due to the resistance of the transformer's copper windings, and to "leakage inductance" caused by flux not linking the primary and secondary coils (coupling coefficient $k < 1$), and to the impedance of the higher voltage (HV, e.g. 10 kV) line and source that feed the transformer's primary windings. You may remember that impedances can be converted to equivalents on the other side of the transformer, using a factor of the *square* of the transformer's voltage-ratio. So even if the 10 kV supply to the transformer has an impedance of 10 Ω, it will only appear to affect the source impedance at the 400 V side of the transformer by about 16 mΩ, and so will typically be negligible compared to the impedances on the LV side. That is one of the useful features of using higher voltages for sending power over longer distances.

Answer 11

The impedance of each conductor from the transformer to the house is

$$Z = R + \mathrm{j}\omega L = 1\,\Omega + 2\pi \cdot 50\,\mathrm{Hz} \cdot 1\,\mathrm{mH} = (1{+}\mathrm{j}0.32)\,\Omega.$$

With no loads connected, our model tells us that there is no current in the line impedances, so the voltages at the house are the same as at the transformer: a balanced 400 V three-phase four-wire set of voltages.

The lamp's current is about $\frac{1}{6}$ A, which is a very small load compared to the other loads we are considering. Let us therefore neglect it: we will find the changes in voltage in phase-a to neutral voltage magnitude, at the house, $|u_{an}|$, with and without each of the other loads. (The amount of change that each load causes may depend on what *other* loads are also connected. But this effect is small as long as the other loads have much higher impedance that the impedance Z of the source.)

a) A 2 kW fan-heater connected between phase-a and neutral.

A heater that gives 2 kW at 230 V has a resistance of $Z_1 = (230\,\mathrm{V})^2 \,/\, 2\,\mathrm{kW} = 26.5\,\Omega$. We'll assume it's reasonable to treat it as purely resistive, and as being constant, although its resistance will change a little with temperature and thus with voltage.

The heater and the lamp share their entire circuit: through the line conductor for phase-a, then back through the neutral. The total impedance in this loop is $2Z$. We can therefore analyse a system of just a Thevenin source whose voltage is that of the transformer a-n terminals, $U_{an} = 230$ V, and whose impedance is $2Z$.

Using voltage division, the voltage across the heater is

$$|u_{an}| = \left|\frac{U_{an} \cdot Z_1}{2Z + Z_1}\right| = \left|\frac{230\,\mathrm{V} \cdot 26.5\,\Omega}{26.5\,\Omega + (2{+}\mathrm{j}0.64)\,\Omega}\right| = 214\,\mathrm{V}.$$

The voltage magnitude has decreased by 16 V, i.e. about 7%.

The above expression can also be used (without the $|\cdots|$ symbol) to find the change in *phase-angle* of the voltage. This is just $-1.2°$. We less often care about this, unless wondering how much an unbalanced load affects the balance of the three-phase voltage.

Notice that the reactance of the line has negligible effect on the result: it is small compared to the sum of line and load resistance, so adding it at 90° makes very little difference, since $(1^2 + (\mathrm{small})^2) \simeq 1^2$.

b) A 3 kW kettle connected between phase-b and neutral.

By assumptions similar to the above, we can define this load as an impedance of $Z_2 = 17.6\,\Omega$.

This current does not pass through phase-a, but it does pass through the neutral. That allows it to affect the lamp's voltage. By voltage division, the voltage across the neutral impedance is

$$u_n = \frac{U_{bn} \cdot Z}{2Z + Z_2} = \frac{\exp(-\mathrm{j}2\pi/3) \cdot 230\,\mathrm{V} \cdot (1{+}\mathrm{j}0.32)\,\Omega}{(2{+}\mathrm{j}0.64)\,\Omega + 17.6\,\Omega} = (-3.00{-}\mathrm{j}11.9)\,\mathrm{V}.$$

The voltage magnitude at the lamp is then

$$|u_{an}| = |230\,\mathrm{V} - (-3.00{-}\mathrm{j}11.9)\,\mathrm{V}| = 233\,\mathrm{V}.$$

The voltage magnitude has *increased* by 3 V when the extra load is connected on another phase.

This is perhaps not too surprising, as the current in the neutral due to the load on phase-b is phase-shifted by more than 90° from the lamp's current. Notice that the change in voltage was mainly at 90° to the original value of u_{an}: the $-$j11.9 V made little difference to the 230 V voltage magnitude, again because of $(1^2 + (\text{small})^2) \simeq 1^2$.

c) A 2.5 kW oven, connected between phase-b and phase-c.

This load has a completely separate current path from the lamp. Its current doesn't pass through either of the impedances that connect the lamp to the source. There is therefore no change in voltage at the lamp. (This assume the transformer is an ideal source; realistically, the electrical and magnetic connections of the transformer permit some influence between the phases; the transformer probably has a Δ-connected primary, meaning that each line current on the HV side is a mixture of line currents on the LV side.)

d) Another 2.5 kW heating element on top of the cooker (spis) is connected between phase-a and phase-b.

This is fairly similar to subquestion 'b': a load takes a current that passes through *one* of the two conductors that affect the lamp. This one conductor is the line conductor of the a-phase, with impedance Z.

Treating this load as $Z_3 = \frac{(400\,\text{V})^2}{2.5\,\text{kW}} = 64.0\,\Omega$, the voltage dropped across the a-phase is

$$u_{ab} = \frac{U_{ab} \cdot Z}{2Z + Z_3} = \frac{e^{j\pi/6} \cdot \sqrt{3} \cdot 230\,\text{V} \cdot (1+j0.32)\,\Omega}{(2+j0.64)\,\Omega + 64.0\,\Omega} = (4.31+j4.65)\,\text{V}.$$

(Hint: the term $e^{j\pi/6} \cdot \sqrt{3}$ is equivalent to $1 - e^{-j2\pi/3}$.)

The voltage magnitude at the lamp is then

$$|u_{an}| = |230\,\text{V} - (4.31+j4.65)\,\text{V}| = 226\,\text{V}.$$

The voltage magnitude at the lamp has decreased by 4 V.

Compare this to the heater, in subquestion 'a'. For both loads, the current passes through the line conductor of phase-a. The heater was a somewhat *lower* power, yet it caused 16 V drop! This line-line connected load gave a much lower voltage drop because: for a given power its current is smaller, as it gets a voltage of 400 V; the current only causes voltage-drop in one of the conductors that the lamp connects to; and the current isn't quite in phase with the open-circuit voltage seen by the lamp (the line-line voltage a-b is shifted compared to the line-neutral voltage a-n) so the phasor describing the drop does not directly subtract from the lamp's voltage.

e) A 1 kW 3-phase motor, during starting.
The rated current is

$$I = \frac{P}{\text{PF} \cdot \sqrt{3}U} = \frac{1\,\text{kW}}{0.95 \cdot \sqrt{3} \cdot 400\,\text{V}} = 1.52\,\text{A}.$$

During starting is it said that the current will be 8 times this level, at a 0.5 lagging power factor. That means 12.2 A, with a phase-shift of $(-\cos^{-1} 0.5) = -60°$ relative to the voltage at the motor's terminals. The power factor during starting is only relevant because the phase-angle of the current affects how much the lamp's voltage changes

The motor will have no neutral current, as it has no neutral connection. Therefore, just the drop that it causes in the phase-a line conductor is relevant to the lamp.

If we make a "first-order approximation" that the voltage at the motor is equal to the voltage at the source, then the a-phase current will be $12.2\underline{/-\pi/3}\,\mathrm{A}$.

This current through the impedance Z causes a voltage drop of

$$(6.10{-}\mathrm{j}10.6)\,\mathrm{A}\cdot(1{+}\mathrm{j}0.32)\,\Omega = (9.50{-}\mathrm{j}8.61)\,\mathrm{V}.$$

The remaining voltage magnitude at the lamp is then

$$|u_{an}| = |230\,\mathrm{V} - (9.50{-}\mathrm{j}8.61)\,\mathrm{V}| = 221\,\mathrm{V}.$$

The lamp's voltage magnitude has decreased by 9 V, but this is only temporary during the short starting-period.

It may seem like crazy cheating that in order to calculate how much the voltage at the load has dropped compared to the voltage at the source, we started by assuming that the voltage at the load was the *same* as at the source. But it doesn't matter much. Looking at u_{an} above, we see a 5% change in magnitude and 2° change in phase-angle, compared to the source. This 5% would be approximately the relative error of the calculated drop: e.g. if we calculate a drop of 10 V we can expect about a 0.5 V error, which doesn't bother us much.

An alternative method would be to represent the motor as an impedance, then use voltage-division again. It's not so obvious with a motor as it was with the earlier heaters, that it really behaves like an impedance when the voltage changes. We were told a specific starting-current: should we assume this is the same when the voltage is lower, or that it scales in proportion to voltage, as it would in an impedance? In the former case, we'd need to work out an impedance based on the actual voltage at the motor terminals, which is what we're trying to find ... a similar problem to the above. Probably the latter assumption is quite reasonable: model the motor using per-phase impedances that take 12.2 A at PF = 0.5 (lag) at 230 V. We use 230 V as this is now a per-phase equivalent, which implies everything being Y-connected. Hence

$$Z_{\mathrm{m}} = 230\,\mathrm{V}\,/\,12.2\underline{/-60^\circ}\,\mathrm{A} = (9.43{+}\mathrm{j}16.3)\,\Omega.$$

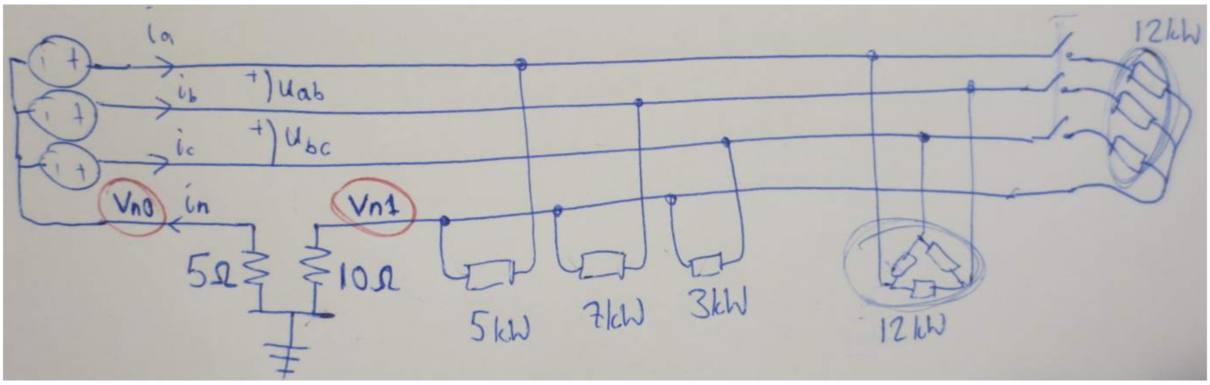
By voltage division, in just the a-phase, the remaining voltage from line to neutral is

$$u_{an} = \frac{U_{an}\cdot(Z+Z_{\mathrm{m}})}{2Z+Z_{\mathrm{m}}} = 230\,\mathrm{V}\cdot\frac{(1{+}\mathrm{j}0.32)\,\Omega + (9.43{+}\mathrm{j}16.3)\,\Omega}{(2{+}\mathrm{j}0.64)\,\Omega + (9.43{+}\mathrm{j}16.3)\,\Omega} = 221\,\mathrm{V}.$$

If we look into more digits of the above solutions (using the same limited precision of numbers as inputs) we see 0.15 V difference.

Exercise 12

The sources in this system form a balanced three-phase source giving line-voltage 400 V.



There are two balanced three-phase loads: one in Δ connection is always connected to the line, and the other in Y connection is connected through a switch. There are three separate single-phase loads, connected from different lines to neutral: these are like an unbalanced Y-connected load.

The neutral conductor has two series-connected resistors in series, with the mid-point shown as zero potential. This models the case where a neutral conductor has broken, and has earth electrodes (jordtag) on both sides of the break, so that neutral currents can flow through the soil. Soil is generally not highly conductive, so the resistance between the two sides of the break will probably be several or many times the resistance along the whole unbroken neutral conductor. We assume here that all four of the conductors – three phases and neutral – have negligible impedance, so we can treat them as nodes.

The voltages $|v_{n0}|$ and $|v_{n1}|$ from the two sides of the neutral to the 'remote earth' (which we've here defined as our reference node) may then be a significant proportion of the supply voltage. The situation is dangerous: pieces of metal that people can touch are often connected back to the neutral conductor for safety(!) reasons, so this conductor should not be allowed to have a dangerous voltage to earth.

We will model the loads as all being pure resistances. For each, the power consumption at its *rated* voltage is given, i.e. the power it would take if connected to the source when the neutral conductor is ideal (no resistance). That isn't the same power as it has when the neutral conductor is properly connected or when the loads are balanced so that the neutral isn't even needed.

a) With the switches open, how large are the potentials $|v_{n0}|$ and $|v_{n1}|$?
You only need consider the potentials' *magnitudes.*
These are the potentials of the neutral conductor at either side of the break, compared to the earth node. Their significance is that these are the voltages someone would be exposed to if contacting the ground (remote from the earth electrodes) at the same time as touching "earthed metal" that is connected to the neutral.

b) How would you expect the above answers to change, if the 12 kW load is switched on?
Calculate and see if this was right.

c) Finally, if the 7 kW and 3 kW loads are changed to be 5 kW each, what will $|v_{n0}|$ and $|v_{n1}|$ become? (Warning: concept-question!)

Answer 12

Let's represent the loads as resistors. We note that the phase voltage is $U = (400\,\text{V})/\sqrt{3} \simeq 230\,\text{V}$.

For the single-phase loads,

$$R_a = \frac{(230\,\text{V})^2}{5\,\text{kW}} = 10.6\,\Omega, \qquad R_b = \frac{(230\,\text{V})^2}{7\,\text{kW}} = 7.55\,\Omega, \qquad R_c = \frac{(230\,\text{V})^2}{3\,\text{kW}} = 17.6\,\Omega.$$

and for the three-phase loads, each resistor has a value

$$R_\Delta = \frac{(400\,\text{V})^2}{(12\,\text{kW})/3} = 40.0\,\Omega \qquad \text{or} \qquad R_\mathsf{Y} = \frac{(230\,\text{V})^2}{(12\,\text{kW})/3} = 13.2\,\Omega.$$

where the subscript denotes whether the resistor is in the delta- or the star-connected load.

As expected, the delta has three time as high resistance for the same load power (remember Δ-Y conversions?). The shown values don't seem to fit that: in fact, $3 \times 13.2 = 39.6$. That's because we just assumed 230 V to be $400\,\text{V}/\sqrt{3}$. To be more accurate we should write 231 V.

We will start by getting solutions in a boring, unthinking, and symbolic way. Later we can as usual show how trivial the practical question is when we're granted a computer program that handles complex numbers. We can also try some elegant tricks, using symmetry and Thevenin equivalents, and less elegant methods of approximation, which is often a useful approach.

a) The switches are *open*. Find $|v_{n_0}|$ and $|v_{n_1}|$.

The star-connected load isn't connected: ignore. The delta-connected load is connected, but it has no connection to the neutral. All currents in this load flow purely within the load and the line conductors and voltage sources, not in the $5\,\Omega$ or $10\,\Omega$ resistors. This load can therefore also be ignored for the purposes of our solution.

Nodal analysis, using the node with the 'earth' symbol as our reference (zero), tells us this:

$$\text{KCL}(v_{n_0}) \qquad \frac{v_{n_0} + U - v_{n_1}}{R_a} + \frac{v_{n_0} + U\,\mathrm{e}^{-\mathrm{j}2\pi/3} - v_{n_1}}{R_b} + \frac{v_{n_0} + U\,\mathrm{e}^{-\mathrm{j}4\pi/3} - v_{n_1}}{R_c} = \frac{-v_{n_0}}{5\,\Omega},$$

$$\text{KCL}(v_{n_1}) \qquad \frac{v_{n_0} + U - v_{n_1}}{R_a} + \frac{v_{n_0} + U\,\mathrm{e}^{-\mathrm{j}2\pi/3} - v_{n_1}}{R_b} + \frac{v_{n_0} + U\,\mathrm{e}^{-\mathrm{j}4\pi/3} - v_{n_1}}{R_c} = \frac{v_{n_1}}{10\,\Omega}.$$

This looks horrible, and the two equations are almost the same.
Putting in numerical values, it comes down to

$$\begin{aligned} 0.481\,\text{S} \cdot v_{n_0} - 0.281\,\text{S} \cdot v_{n_1} &= \mathrm{j}15.0\,\text{A} \\ 0.281\,\text{S} \cdot v_{n_0} - 0.381\,\text{S} \cdot v_{n_1} &= \mathrm{j}15.0\,\text{A} \end{aligned}$$

where the unit S is 'siemens', equivalent to Ω^{-1}.
This solves as $|v_{n_0}| = 14.4\,\text{V}$ and $|v_{n_1}| = 28.7\,\text{V}$.

b) Now the switches are *closed*. Find $|v_{n_0}|$ and $|v_{n_1}|$.

The Y load is now connected. It is a 4-wire load, so its star-point connects to the neutral that the single-phase loads also connect to. Will this extra load make more of a voltage drop across the break (the total $15\,\Omega$ resistance)? That doesn't seem very reasonable, as it's a balanced load, so we'd not expect it to have had any neutral current by itself. Will it have no effect? After all, it's balanced.... Or will it somehow reduce the voltage drop? In fact, the last of these is true! One way to see it is that any unbalance added to a balanced load (e.g. a further parallel resistance added to one phase) will move the neutral-point voltage so that that phase has less voltage across it and the others have more: consequently, among the balanced impedances, the one in parallel with the extra load carries less current than usual, and the others carry more than usual, partly compensating for the extra current in the unbalanced load.

To analyse this situation, we can repeat 'a', but with a further resistance of $R_{\mathsf{Y}} = 13.2\,\Omega$ connected in parallel with each of R_a, R_b and R_c. In the numeric solution this modifies the coefficients of v_{n_0} and v_{n_1}, but does not affect the j15.0 A at the right hand side. We get

$$\begin{aligned} 0.706\,\mathrm{S} \cdot v_{n_0} - 0.506\,\mathrm{S} \cdot v_{n_1} &= \mathrm{j}15.0\,\mathrm{A} \\ 0.506\,\mathrm{S} \cdot v_{n_0} - 0.606\,\mathrm{S} \cdot v_{n_1} &= \mathrm{j}15.0\,\mathrm{A} \end{aligned}$$

This solves as $|v_{n_0}| = 8.73\,\mathrm{V}$ and $|v_{n_1}| = 17.5\,\mathrm{V}$. Indeed, the extra load made the situation better.

c) Keep the switches closed. Make all three single-phase loads be 5 kW. Find $|v_{n_0}|$ and $|v_{n_1}|$.

Here, the loads are all balanced.
There is therefore no neutral current.
With $i_n = 0$, we have $v_{n_0} = v_{n_1} = 0$.

Some nicer methods.

For a start, we didn't really need to write two almost identical KCLs, with two unknowns. One way to avoid it, while still following standard steps of nodal analysis, is just to **redefine** our reference potential. The node under the $10\,\Omega$ and $5\,\Omega$ resistors represents the 'remote earth' (the potential of the earth's surface far from our electrodes) — but that doesn't mean we have to define this as our 'earth' for doing circuit analysis; we can choose anything. Let's choose the neutral of the source as our reference: $v_{n_0} = 0$. Then **combine** the two resistors to $15\,\Omega$, and temporarily forget the node between them: we get one KCL equation, with just one unknown (which is $v_{n_1} - v_{n_0}$). Solve for this, then use voltage division between the resistors to find the potential at the point between them. The requested voltage magnitudes between this middle point and the two sides are then easily found. We use this method in the numerical example below.

Looking for **symmetry**, our three different load resistors could be changed into a balanced Y-load of three resistors all of value R_c (the smallest, i.e. highest resistance, of the unbalanced single-phase loads), along with two further resistors, from phase-a and phase-b to neutral. These further two must be chosen so that when in parallel with a resistance R_c they form the same resistance as in the real circuit, R_a or R_b. Then the solution we find is *more general* as we could directly extended it to the case when the switch is closed, by making the balanced part of our load have a lower resistance. By this approach we have 'broken out' all the symmetry that we can find, leaving the asymmetric part as simple as possible.

One sort of **approximation** that could be useful is to assume that the voltage dropped across the two earthing-electrode resistances ($5\,\Omega + 10\,\Omega$) is 'small' so that the current i_n is fairly similar to what it would be if the neutral were not broken. This current is easily calculated, because the voltage across each load is then immediately known. Having found this current, we use Ohm's law to find what $|v_{n_0}|$ and $|v_{n_1}|$ it implies. If they do sum to a small amount (e.g.

not more than perhaps 1/20 of the phase-voltage of the source) then our result is a moderately good approximation for assessing the safety. If it's more, then we might want to calculate more accurately. In the particular case, if it's more than a few tens of volts, then it's a dangerous situation anyway.

When we have a balanced 4-wire Y-connected load as well as an unbalanced one, a neat Thevenin method can be used. That would be the case when the switched load in our circuit is connected, or if one uses the above symmetry-method to split the single phase loads into a balanced and unbalanced three-phase load. Find the Thevenin equivalent between the star-point of the balanced load and neutral of the source, *without* the neutral conductor connecting them. This equivalent is $U_{\mathrm{T}} = 0$ due to the balanced sourec and load, and $R_{\mathrm{T}} = R_{\mathsf{Y}}/3$ because all three of these resistors are seen in parallel (think of zeroing the voltage sources). So, without affecting the potentials we are seeking, we can replace all the balanced Y-connected 4-wire loads with a single resistor $R_{\mathsf{Y}}/3$ connecting "across the break", i.e. between the points v_{n_0} and v_{n_1}, as these are the nodes of the source neutral and the balanced-load neutral. This might simplify a single analysis. But its main advantage, just like using equivalents for studying dividers and time-constants, is that it helps us estimate what would happen if various aspects of the circuit were changed, more easily than from the original circuit.

Solutions can be found **numerically**, which is particularly easy to write if the above 'redefinition' is used to **combine** the two series resistors.

With the switches open, we confirm the result from subquestion 'a'.

```
U = 400/sqrt(3);  % we use phase-voltage more often in this question
Ua=U; Ub=Ua*exp(-1j*2*pi/3); Uc=Ub*exp(-1j*2*pi/3);
Rs = U^2/(12e3/3);  Ra = U^2/5e3; Rb = U^2/7e3; Rc = U^2/3e3;
R0 = 5;  R1 = 10;
v1mv0 = ( Ua*(1/Ra) + Ub*(1/Rb) + Uc*(1/Rc) ) ...
      / ( 1/Ra + 1/Rb + 1/Rc + 1/(R0+R1) )
v0 = abs( - v1mv0 * R0/(R0+R1) )
v1 = abs( v1mv0 * R1/(R0+R1) )
```

This is easily modified for further Y-connected balanced loads. For example, let's try it with the switches closed, to check the result from subquestion 'b'.

```
U = 400/sqrt(3);  % we use phase-voltage more often in this question
Ua=U; Ub=Ua*exp(-1j*2*pi/3); Uc=Ub*exp(-1j*2*pi/3);
Rs = U^2/(12e3/3);  Ra = U^2/5e3; Rb = U^2/7e3; Rc = U^2/3e3;
R0 = 5;  R1 = 10;
v1mv0 = ( Ua*(1/Ra + 1/Rs) + Ub*(1/Rb + 1/Rs) + Uc*(1/Rc + 1/Rs) ) ...
      / ( 1/Ra + 1/Rb + 1/Rc + 3/Rs + 1/(R0+R1) )
v0 = abs( - v1mv0 * R0/(R0+R1) )
v1 = abs( v1mv0 * R1/(R0+R1) )
```

We could have tried to be 'clever' by not even writing the terms `1/Rs` in the expression for `v1mv0`; after all, we know that they cancel, since the three voltages `Ua,Ub,Uc` are balanced. But if the program might be used more than just this once, there's a risk that we later change it to make an unbalanced source, forgetting that the equations contain a simplification based on the assumption of a balanced source! It's quite nice to keep the generality, or at least to comment clearly about the assumption.