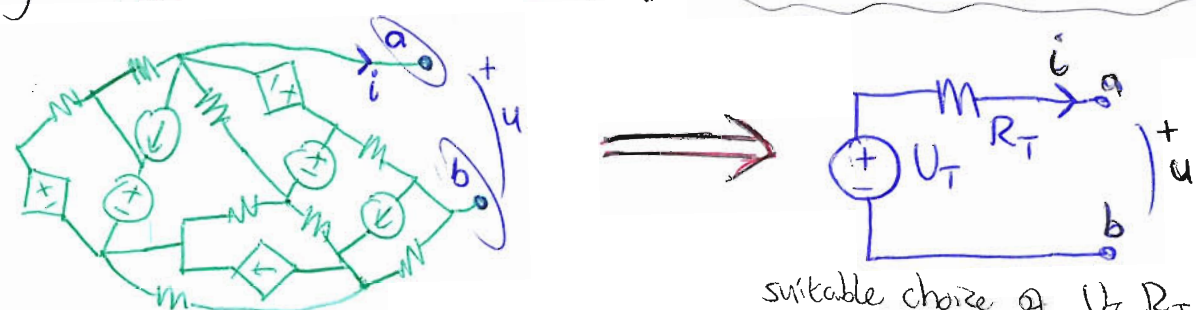
TWO-TERMINAL EQUIVALENTS and MAXIMUM POWER

tvåpol ekvivalent, Helmholtz–Thevenin teorem   maximal effektöverföring

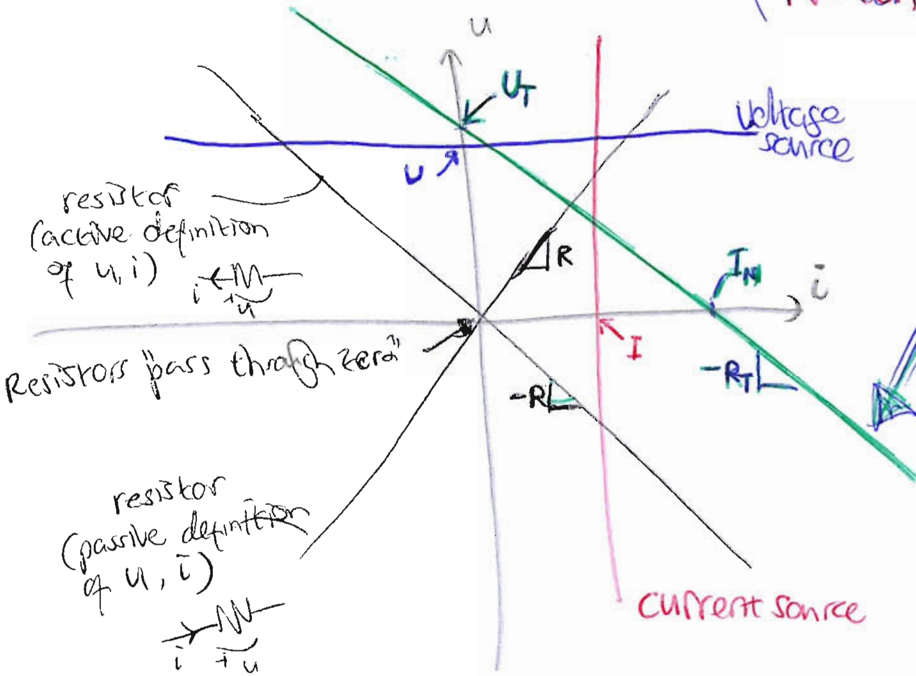
A very deep, useful, "powerful" circuit theorem is that any linear dc circuit's behaviour at two terminals (relation of u & i) can be equivalently modelled by a Thevenin or Norton source!

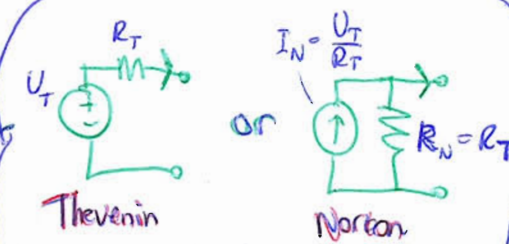
And, for a Thevenin source (and thus for a Norton source or any two terminal dc linear circuit) one choice of voltage or current or resistance at its terminals results in maximum possible power output.



suitable choice of U_T, R_T makes this equivalent in its u, i relation

Another way of describing the {Thevenin equivalent / or Norton equivalent} concept!





ANY straight line in u,i plane

IGNORE THIS ↑ if you don't like diagrams!

"the u,i relation for any linear 2-terminal dc circuit is a straight line"

(So how do we work out what line, from the circuit?)

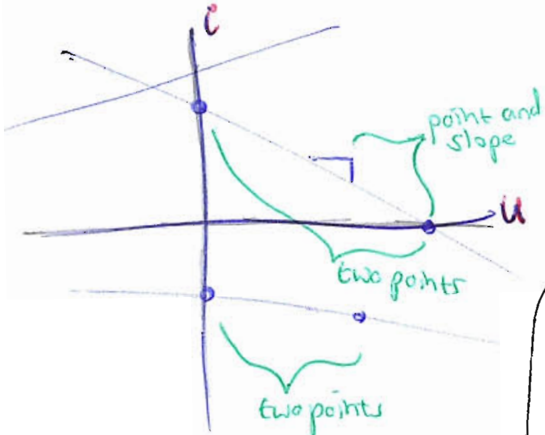
pause: why is it useful to find a $\begin{Bmatrix}\text{Thevenin} \\ \text{Norton}\end{Bmatrix}$ equivalent ?

Lots of reasons! (it really is useful) -- here are a few ---

- simplify all of a big circuit except for a little part that we are designing/changing/studying --- then our analysis is with just our little part and a Thevenin/Norton source
- conceptual understanding: by seeing how the equivalent changes (eg. R_T changes) when we change something in the original circuit, we see how we can expect other things connected to the circuit's terminals to be affected.
- practical connections — the idea of output resistance (amplifier) or source impedance (power) is basically Thevenin resistance; we can determine it by measurement as well as calculation

How can we find the values of U_T R_T (or I_N R_N) to make a Thevenin or Norton equivalent of a given circuit?

There are several choices:



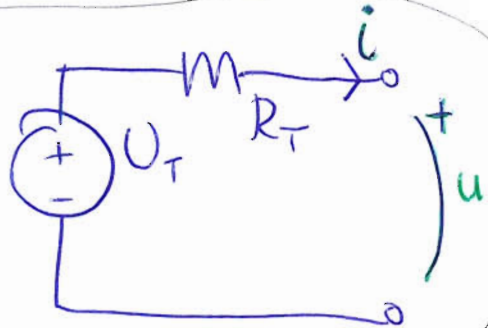
we have to define a line in the u,i plane ... two pieces of information are needed: two points, or point and slope

(2 koordinater, eller koordinat och lutning gradient)

Examples of common approaches

- derive a $\boxed{u = \text{something} + \text{something}\; i}$ equation for the circuit, and compare to the Thevenin/Norton equation
- find shortcircuit current and open circuit voltage (often simpler circuits to solve)
- find one of the above, and find slope directly

Remember u, i relation for Thevenin & Norton sources



$$u = U_T - R_T\, i$$

or

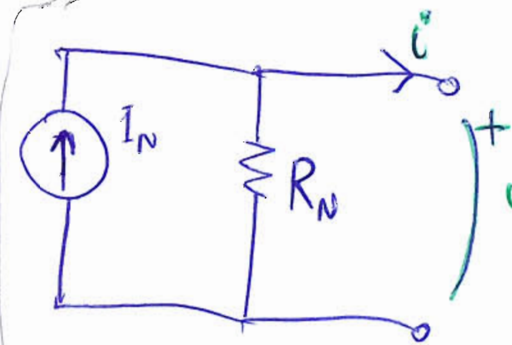
$$i = \frac{U_T}{R_T} - \frac{1}{R_T}\, u$$

$\longleftrightarrow$

for equivalence,

$R_T = R_N$

$U_T = I_N R_N$



$$u = I_N R_N - R_N\, i$$

$$i = I_N - \frac{1}{R_N}\, u$$

Try deriving all of those!
(KCL, KVL, Ohm)

Notice the form of all these equations: $y = mx + c$

straight line

One method direct derivation of u,i relation

- sometimes hard algebra, better by other methods ☹
- sometimes very convenient! ☺
- no limitations (other than difficulty) — works with dependent sources

might not be easy!

- Get an equation relating u and i for the circuit:

wild example:

$$u = \frac{(I_7+I_5)R_3R_4}{R_2+R_1} + \frac{U_2 R_6}{R_7+R_8} - \frac{R_1 R_2}{R_2+R_3}\cdot(1+k)\, i$$

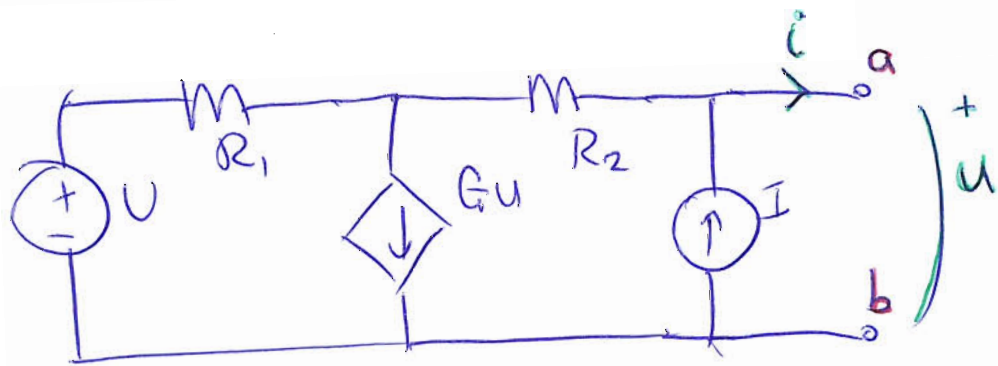
- Compare it to the equation for a Thevenin or Norton source, and thus find the values for the equivalent

$$u = U_T - R_T\, i \qquad \text{(Thevenin source)}$$

(Notice $u = U_T - R_T i$ or $u = U_T + R_T i$ depends on chosen definition of u & i)

Example direct derivation (Quite a hard example)

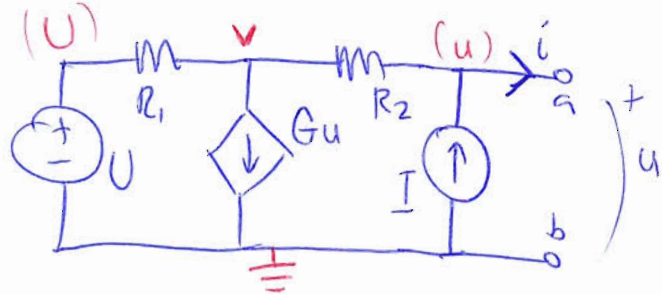
find the Thevenin equivalent of this circuit, seen between a-b.



Hmmm. Could try superposition, source transformation, ...?

Let's go for nodal analysis ... often a good choice if no really good simplification is obvious. By supernode or "simplified branch" we only need two equations and unknowns here

cont.d. define a reference node and a potential v.
the other potentials are either known (U) or marked already (u).



Want to eliminate V, to have just u and i unknown, in a single equation

KCL(v) $$\frac{v-U}{R_1} + Gu + \frac{v-u}{R_2} = 0$$

$$\Downarrow$$

$$v\left(\frac{1}{R_1}+\frac{1}{R_2}\right) + Gu - \frac{u}{R_2} - \frac{U}{R_1} = 0$$

KCL(u) $$\frac{u-v}{R_2} - I + i = 0$$

$$\Downarrow$$

$$u - v = -R_2(-I+i)$$

$$v = R_2(i-I) + u$$

substitute this expression for v

contd.

$$\left(R_2(i-I)+u\right)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+Gu-\frac{u}{R_2}-\frac{U}{R_1}=0$$

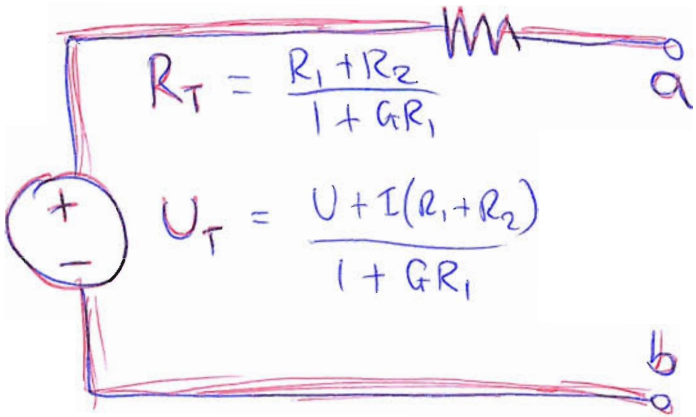
collect terms:

$$u\left(\frac{1}{R_1}+\cancel{\frac{1}{R_2}}-\cancel{\frac{1}{R_2}}+G\right)-\frac{U}{R_1}-IR_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+iR_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)=0$$

$$u=\frac{\frac{U}{R_1}+I\cancel{R_2}\left(\frac{R_1+R_2}{R_1\cancel{R_2}}\right)-i\cancel{R_2}\left(\frac{R_1+R_2}{R_1\cancel{R_2}}\right)}{\frac{1}{R_1}+G}$$

$$u=\frac{\frac{U}{\cancel{R_1}}+\frac{I}{\cancel{R_1}}(R_1+R_2)-\frac{i}{\cancel{R_1}}(R_1+R_2)}{\frac{1+GR_1}{\cancel{R_1}}}=\underbrace{\frac{U+I(R_1+R_2)}{1+GR_1}}_{U_T}-\underbrace{\frac{R_1+R_2}{1+GR_1}}_{R_T}i$$

The Thevenin equivalent of the original circuit, between a-b, is therefore:



$$R_T = \frac{R_1 + R_2}{1 + GR_1}$$

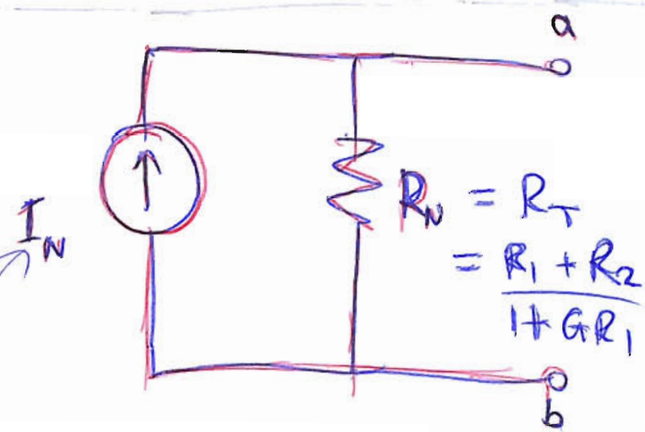
$$U_T = \frac{U + I(R_1 + R_2)}{1 + GR_1}$$

(Careful to set the direction right, with respect to the terminals ... not upside down source)

If we wanted a Norton equivalent, we could use source transformation:

$$I_N = \frac{U_T}{R_T} = \frac{U + I(R_1 + R_2)}{R_1 + R_2}$$

$$\therefore \; I_N = \frac{U}{R_1 + R_2} + I$$



$$R_N = R_T = \frac{R_1 + R_2}{1 + GR_1}$$

In some cases, calculating $\frac{U_T}{R_T}$ or $I_N R_N$ may be harder (in algebra),

So we might benefit from looking for an expression in the form $i = I_N - \frac{u}{R_N}$ at the start, (instead of for $u = U_T - R_T i$) if we want a Norton equivalent (instead of Thevenin).

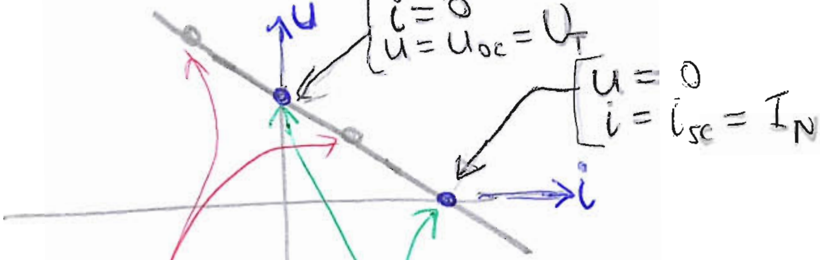
But we might instead find it's easier to get one form of the equation then convert by source transformation ... depends on the circuit!

A note: in that last example we had 3 unknowns & 2 equations at the start .. is that ok?

Yes: we are trying to get a relation between u and i, not to solve every unknown ... we want one equation to remain.

This is expected, because we haven't defined the circuit fully. There is a "something" connected at a-b, and until we define that something we can't solve everything.

Another method: two points ... e.g. $\begin{cases} \text{open circuit voltage } u_{oc} \\ \text{short circuit current } i_{sc} \end{cases}$

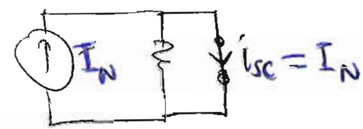


any two points define the line so that we could find I_N U_T R_T

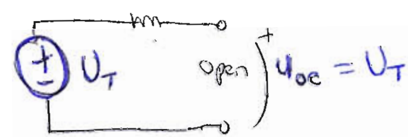
but these two points are a very nice choice!

Why? ...

If we find i_{sc}, this gives us I_N directly:



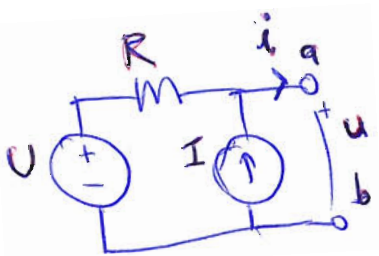
And u_{oc} would give U_T directly:

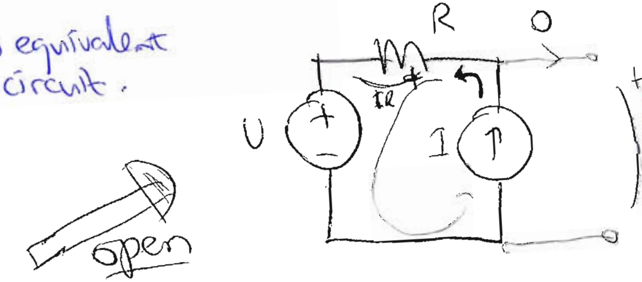


And we know that $R_N = R_T = \dfrac{u_{oc}}{i_{sc}}$
(source transformation)

A further advantage of choosing the open circuit voltage and short circuit current is that adding shorts or opens can make the circuit simpler to solve than with a generic "u, i" relation. (The down-side is that two different circuits have to be solved.)

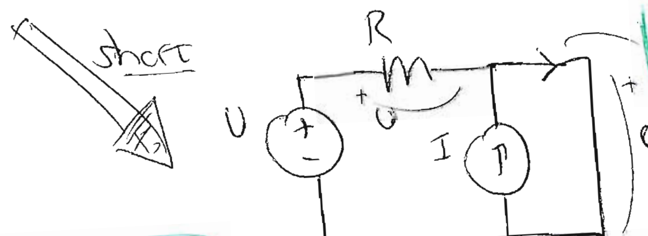
example: find Norton equivalent of this circuit.





$$\boxed{u_{oc} = U + IR}$$

simple KVL



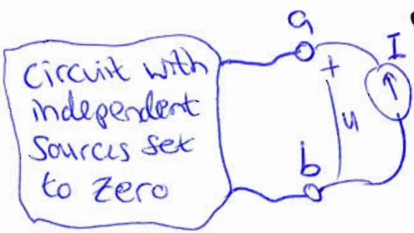
$$\boxed{i_{sc} = \frac{U}{R} + I}$$

So $\boxed{R_N = \frac{u_{oc}}{i_{sc}} = \frac{U+IR}{\frac{U}{R}+I} = R}$

Another method involves finding <u>either</u> u_{oc} <u>or</u> i_{sc}, then finding the resistance directly.

This is based on superposition; set all independent sources to zero, and see how the circuit behaves at its terminals.

General case
(possible dependent sources)



apply a current I

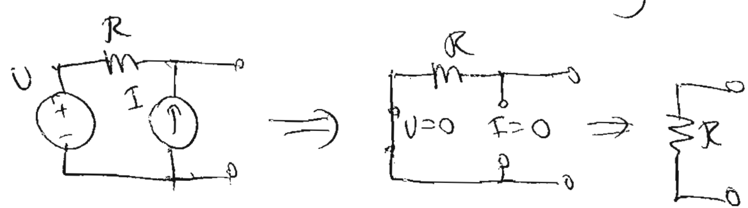
then measure (calculate) voltage u, and

$$R_T = \frac{u}{I}$$

We won't pay much attention to this – probably focus on the short/open method

Very useful case, with no dependent sources

Set sources to zero, and find equivalent resistance!
It's only resistors remaining...



$\therefore R_T = R$ (compare to earlier solution)