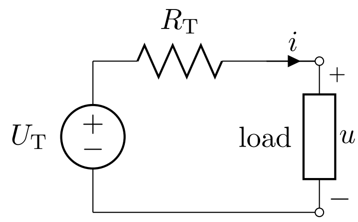
Maximum power transfer

[This is copied from the old 'chapter', as it seems already sufficiently short and clear.]

A question of surprisingly frequent interest is what maximum power can be *obtained from* some circuit that has two terminals. (In our ideal circuits we consider just the limitation caused by the circuit's u-i relation, not other practical limitations such as whether a wire would melt, etc.)

We've seen that our linear dc circuits with two terminals can be modelled by Thevenin or Norton equivalents. So let's study the power available from a Thevenin source, with its terminals connected to some generic component that we'll call the *load* (sv: *last*). Our results will then be general to the power available from any two-terminal linear dc circuit. It is common to assume the load to be a resistor whose value we can chose; but we start with the more general case where the load could be a voltage or current source, or even another circuit with a Thevenin equivalent model!



The question is what power can be got *from* the Thevenin circuit shown at the left of the terminals, *to* this other two-terminal thing that we call the load.

A Thevenin equivalent, with finite positive resistance, *does* have a maximum power that it can supply out of its terminals. Nothing that is connected at the terminals can extract more power than this. (The same applies, of course, to a Norton equivalent.)

Consider the behaviour of the Thevenin equivalent in the u-i plane. From the relation

$$u = U_T - iR_T$$

the power out from the terminals *to* the load can be expressed as a function of the current,

$$P = ui = U_T i - i^2 R_T,$$

which can be seen to be the power provided by the voltage source U_T, minus the power consumed by the Thevenin resistance R_T.

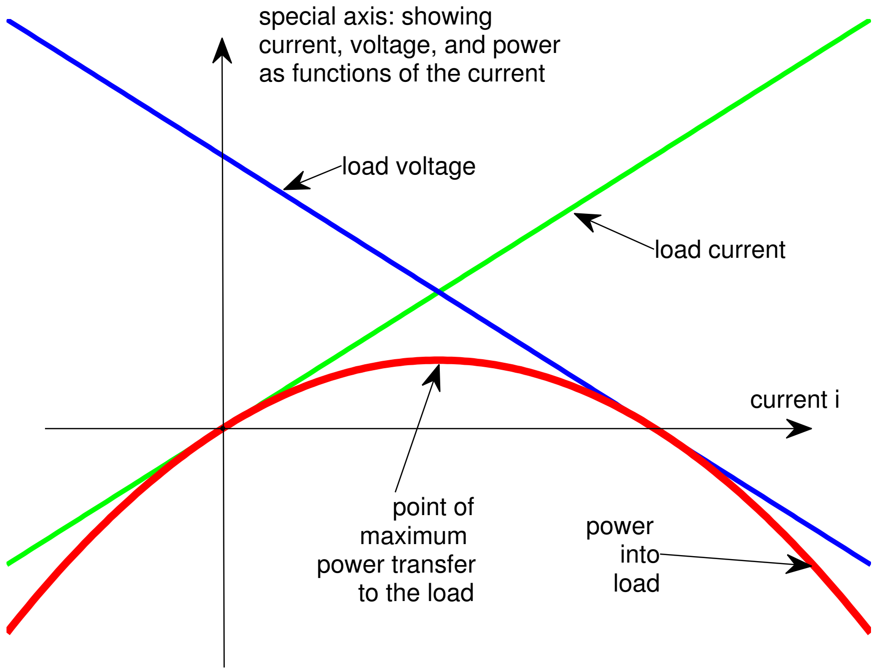
If the load is like an open-circuit, taking no current from the source, $i = 0$, then $P = 0$: there is no power into or out of the load. In this case, there is no power in or out of any of the three components.

If the load is like a short-circuit, taking so much current that there is zero voltage at the terminals, $u = 0$, then there is also no power transfer between the Thevenin equivalent and the load. In this case, all the power out of the Thevenin equivalent's voltage source is then lost in the Thevenin equivalent's resistance.[1]

If we force even more current through the circuit, so that $i > i_{sc}$, by having a 'load' that is a voltage or current source that helps the current to flow, then power flows *into* the Thevenin equivalent.

[1] For amusement, notice a difference between the Thevenin and Norton cases: it is in *open*-circuit conditions that the Norton equivalent's current source is feeding a high power to the equivalent's resistance. We only require that the two types of source behave in the same way at their terminals: the internal power consumptions are neglected.

These situations are shown in the following figure, where the horizontal axis shows the current i. The vertical axis shows three different variables, that are plotted against current in the three curves.



The voltage u decreases from the Thevenin equivalent's open-circuit value at $i = 0$, to zero when $i = i_{\rm sc}$. A line showing the current i is marked as well: this is trivial as it is just a plot of i versus i ... its purpose is to make clearer how the *product* of the u versus i and i versus i lines gives the quadratic curve of P versus i.

This curve of P, the power delivered to the load, shows that at the open-circuit and short-circuit conditions there is zero power transfer between the Thevenin equivalent and the load, and at currents *between* these two points there is a power transfer from the Thevenin equivalent to the load, with a maximum point in the middle.

For currents outside this range, the power flow is *into* the Thevenin equivalent, which indicates that the load in such cases cannot be a resistor (not a positive one, anyway)! There is no limit to how much power can be pushed *into* the Thevenin equivalent. The limit on the power *out* comes from the Thevenin resistance consuming the power that the Thevenin voltage-source generates: the power lost in the Thevenin resistance increases quadratically with current.

How can we find the current needed for maximum power transfer to the load, without just guessing by looking at the curve? The *maximum point* corresponds to a *zero gradient*, which means that the derivative of P versus i curve must be zero at the point where maximum power is obtained. (We can see from the curve that the function $P(i)$ has a maximum, not a minimum.) This derivative is

$$\frac{\mathrm{d}P}{\mathrm{d}i} = \frac{\mathrm{d}}{\mathrm{d}i}\left(U_{\rm T} i - i^2 R_{\rm T}\right) = U_{\rm T} - 2iR_{\rm T},$$

which has its zero point when

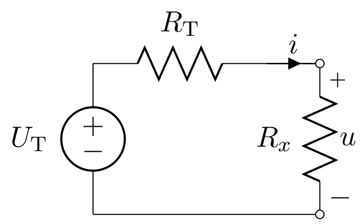
$$i = \frac{U_{\mathrm{T}}}{2R_{\mathrm{T}}}.$$

Thus, the maximum possible power is obtained from the Thevenin equivalent when its current half of its short-circuit. You should also be able to show that the voltage u at this point is half of the open-circuit voltage.

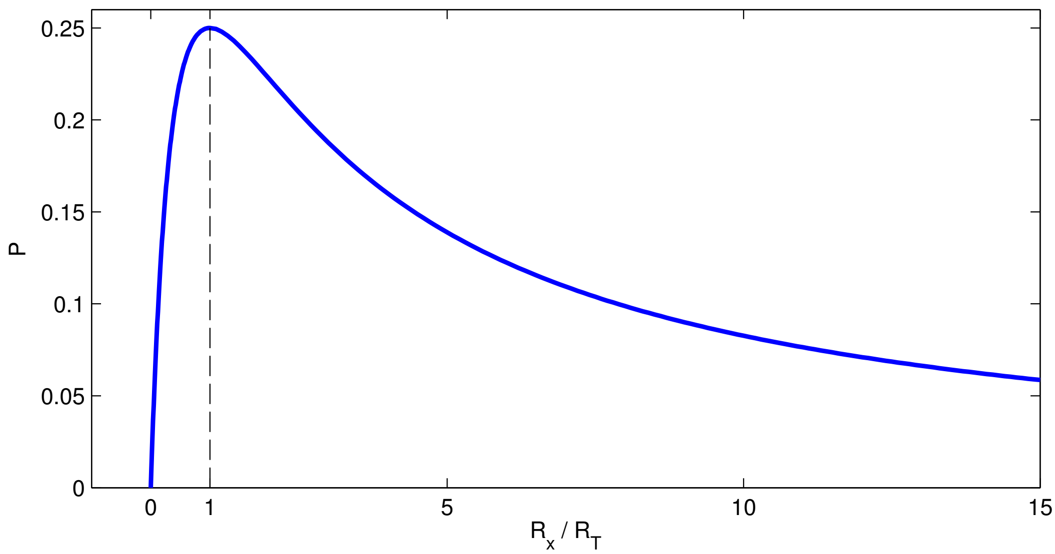
Maximum power to a resistor

All the above was based on a load of "something" connected to the two terminals of a Thevenin equivalent; this is the general case of getting maximum power, where we even consider negative currents and currents greater than the Thevenin equivalent's short-circuit current.

More traditionally, *maximum power transfer* considers the load to be just a resistor, as with R_x in the following circuit. We can vary the resistance to achieve maximum power transfer to the load, from a given Thevenin equivalent. In this more limited case, negative powers are not possible, as a resistor cannot provide power.



When the resistor is varied from zero to a huge resistance, the situation changes from short-circuit to open-circuit conditions for the Thevenin equivalent. This corresponds to the range of currents for which positive powers were transferred to the load, in the previous Section. The following figure shows the power transferred to the load resistor R_x, as a function of R_x/R_{T}.



This was calculated by

$$P = R_x i^2 = R_x \left(\frac{U_{\mathrm{T}}}{R_{\mathrm{T}} + R_x} \right)^2 .$$

In the previous Section, the maximum power transfer was found to occur when the current to the load was $i = \frac{i_{sc}}{2} = \frac{U_T}{2R_T}$. When the load is a resistor, this current will occur when the total circuit resistance is twice the Thevenin resistance; thus, the maximum power criterion is that

$$R_x = R_T.$$

This is the way that the maximum power transfer theorem is usually expressed. Deriving it directly for R_x involves a rather uglier derivative, and does not show that it is a more general concept than just resistive loads.

Who cares?

The interest in maximum power arises in many cases.

In electronic design, one might want to choose a load that will extract the maximum possible signal power from a source of known Thevenin resistance. Or one might want to know the worst case of how much power would be able to put into a variable resistor by the rest of the circuit that the resistor is connected to, in order to check that the resistor won't get burned.

In electric power engineering there is a wide use of Thevenin equivalents to model the rest of the power system that is hidden behind the 230 V socket outlet, or behind the 400 kV busbars in a substation. In some highly stressed cases in the high-voltage system, the loading can approach the maximum power transfer conditions, and some types of load then try to take even more current, leading to "voltage collapse".

Little details for the interested: negative source-resistance

We have assumed in the above that the source and load have *positive* resistance, which is expected for a passive circuit where resistance is e.g. a power loss in a poor conductor. In contrast, some circuits with dependent sources (that model transistors, amplifiers, etc) may instead have negative source resistance where the voltage increases with increased current.

See the *Extra* section in the Chapter-text if you'd like to think about some subtleties of putting power into a Thevenin circuit, or the behaviour of a Thevenin source with negative source resistance. As the name 'Extra' suggests, this is not obligatory material!