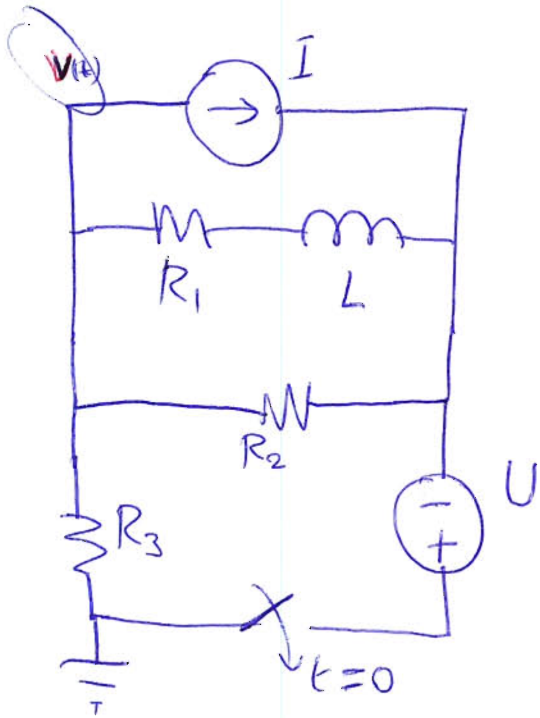
TIME FUNCTIONS (cont.d)

We continue with the "first-order, constant forcing-function" circuits
(only one capacitor or inductor; sources constant except for step/switch at start)

Previously we aimed to find a differential equation, and solve it.
(we noted that a good safe choice is to use the continuous variable of the capacitor or inductor as the equation's dependent variable)

* Now we consider converting all of the circuit <u>except</u> the inductor or capacitor into a Thevenin or Norton equivalent, then solving this simpler case.
* We then consider a "quick cheat" method (for these "first order" circuits) of finding {initial value, final value, time constant} and plugging these into a solution.

Does this strike terror into the heart?

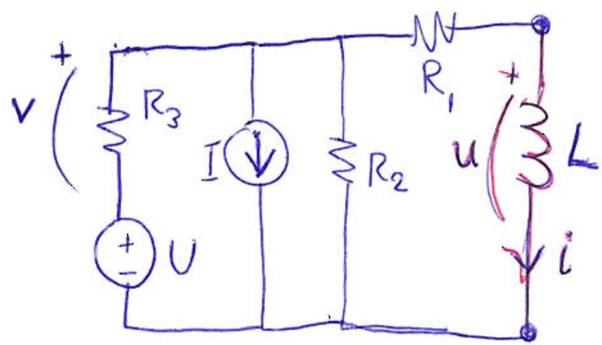


find $v(t)$ for $t > 0$

It probably should cause worry.
It gives nastier expressions than we would normally expect in an exam.
(It would be much neater if we were using numeric values and a computer)

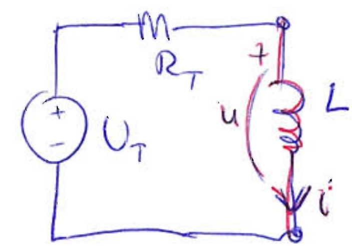
As usual ... when worried, step back, think of breaking it down, follow established rules you've seen before, ... and redraw (if useful) !

1) Let's start by redrawing for $t \geq 0$
We'll do this in a way that clearly separates the inductor from the rest of the circuit.

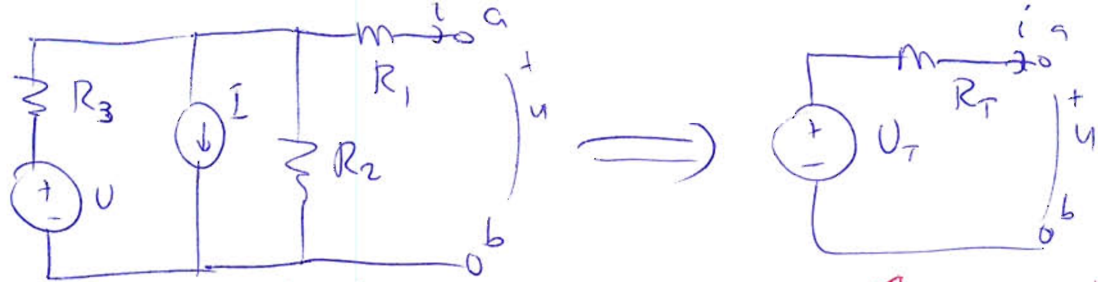


2) Now we'll solve for the continuous variable $i(t)$.

To break the work into easier parts, let's first convert the circuit to Thevenin form, for which we can more easily solve $i(t)$.



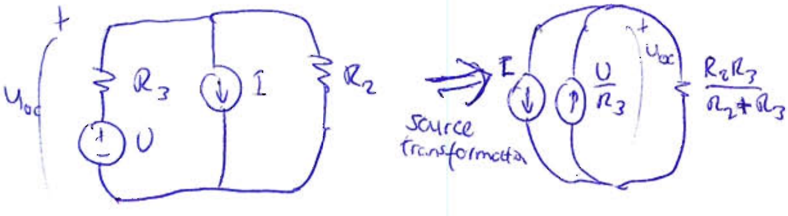
$t \geq 0$



original

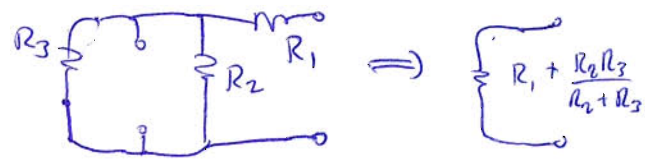
Thevenin equivalent

U_T: (solve open-circuit)



$$u_{oc} = U_T = \frac{\left(\frac{U}{R_3} - I\right) R_2 R_3}{R_2 + R_3} = \frac{UR_2 - IR_2R_3}{R_2 + R_3}$$

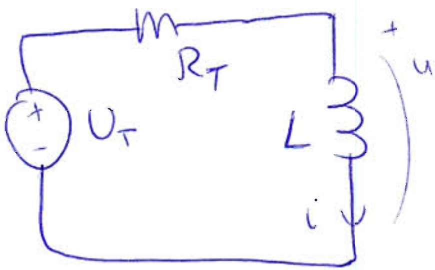
R_T: no dependent sources, so easily found by "set all sources to zero"



$$R_T = R_1 + \frac{R_2R_3}{R_2+R_3} = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_2 + R_3}$$

for the Thevenin circuit,

$(t \geq 0)$



KVL: $U_T = iR_T + \frac{di}{dt} L$

$$\frac{di}{dt} + \frac{R_T}{L} i = \frac{U_T}{L}$$

$$\boxed{i(t) = \frac{U_T \cancel{/L}}{R_T \cancel{/L}} + k e^{-\frac{R_T}{L}t}}$$

to be determined!

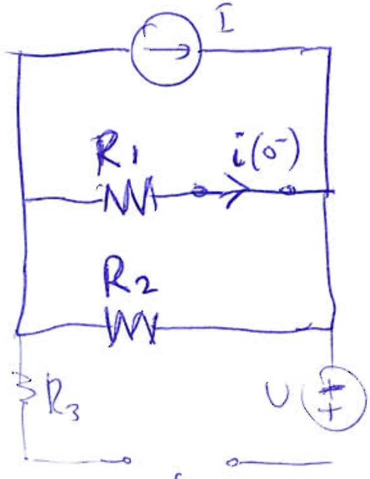
How do we find the initial condition?

The Thevenin circuit was only derived for $t \geq 0$, not $t < 0$ (it would in our case be a different U_T and R_T then)

We could make a different Thevenin circuit --- or else just solve the equilibrium directly.

vgv

In the equilibrium at $t = 0^-$, find the inductor's current



open at $t < 0$

$$i(0^-) = -I \frac{R_2}{R_1 + R_2}$$

by continuity $i(0^+) = i(0^-)$

So at $t = 0^+$, use this initial condition to find k

$$i(0^+) = -I \frac{R_2}{R_1 + R_2} = \frac{U_T}{R_T} + k \overbrace{e^{-\frac{t R_T}{L}}}^{\approx 1 \text{ as } t=0}$$

Now is the time to insert the real U_T e R_T values.

$$\frac{U_T}{R_T} = \frac{\dfrac{UR_2 - IR_2R_3}{\cancel{R_2+R_3}}}{\dfrac{R_1R_2 + R_1R_3 + R_2R_3}{\cancel{R_2+R_3}}} = \frac{UR_2 - IR_2R_3}{R_1R_2 + R_1R_3 + R_2R_3}$$

so $$i(0^+) = \frac{U_T}{R_T} + k \Rightarrow k = \frac{-IR_2}{R_1+R_2} - \frac{UR_2 - IR_2R_3}{R_1R_2 + R_1R_3 + R_2R_3}$$

Put this into the time function:

$$\boxed{i(t) = \frac{UR_2 - IR_2R_3}{R_1R_2 + R_1R_3 + R_2R_3} - \left(\frac{UR_2 - IR_2R_3}{R_1R_2 + R_1R_3 + R_2R_3} + \frac{IR_2}{R_1+R_2}\right) e^{-t/\tau}}$$

$[t \geq 0]$ important limitation!

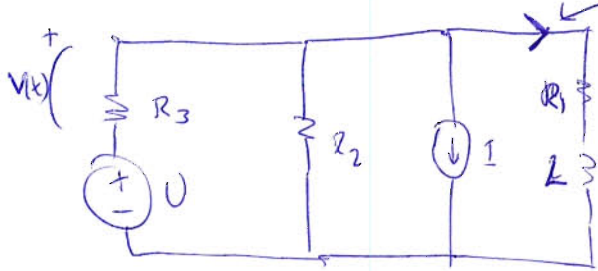
where $\tau = \dfrac{L}{R_T} = \dfrac{L(R_2+R_3)}{R_1R_2 + R_1R_3 + R_2R_3}$

We can <u>check</u> this solution in several ways:

- dimensional analysis
- does it give the expected initial (0^+) and final ($\to \infty$) values?
- solve in a different way, and compare
  e.g. write an ODE directly, without the Thevenin equivalent
  In our circuit a single KCL would be a good starting point.

But we're still not finished!

We were supposed to find a potential $U(t)$ $(t>0)$, which is the voltage across R_3



$i(t)$ that we solved on the previous page.

KCL $$\frac{V}{R_3} + \frac{U+V}{R_2} + I + i = 0 \Rightarrow V(t) = \frac{-UR_3 - (I+i)R_2R_3}{R_2+R_3}$$

$$V(t) = \frac{-UR_3 - IR_2R_3 - \left(\frac{UR_2 - IR_2R_3}{R_1R_2+R_1R_3+R_2R_3}\right)R_2R_3}{R_2+R_3} + \frac{R_2R_3}{R_2+R_3}\left(\frac{UR_2 - IR_2R_3}{R_1R_2+R_1R_3+R_2R_3} + \frac{IR_2}{R_1+R_2}\right)e^{-t/\tau}$$

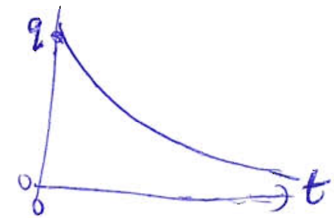
horrible! (and do check for the algebra, critically). But the <u>method</u> was the main point!

In both of these solutions ($v(t)$ and $i(t)$) we saw a general feature/form of the time function:

$$\boxed{f(t) = \underbrace{p + q}_{\text{constants}}\, e^{-t/\tau}}$$

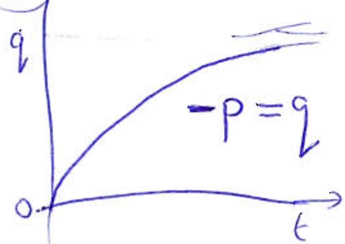
function, such as v or i

The $q e^{-t/\tau}$ is a decaying exponential:

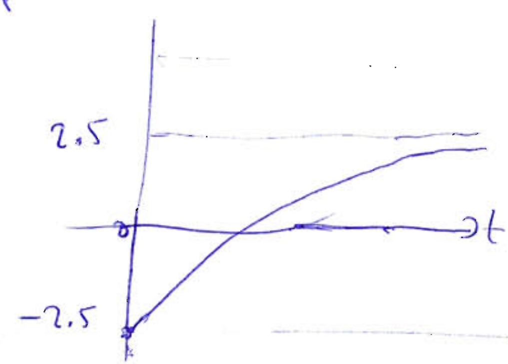


We get various possible $f(t)$ from our circuits, **all using this basic shape** (if they're not "forbidden cases" with impulses or constant rise)

e.g.



or



$p = 2.5$
$q = -5$

highly relevant to lab 3!

We can instead think of an (instead of the full ODE solution)

- initial value $f(0^+)$
- and final value $f(\infty)$
- and time constant τ

Then to build our exponentially changing function from $f(0^+)$ to $f(\infty)$,

$$f(t) = \underbrace{f(\infty)}_{\text{final value}} + \underbrace{\left(f(0^+) - f(\infty)\right)}_{\text{difference between final and initial}} \underbrace{e^{-t/\tau}}_{\text{how quickly the transition happens}}$$

<u>How to find values:</u>

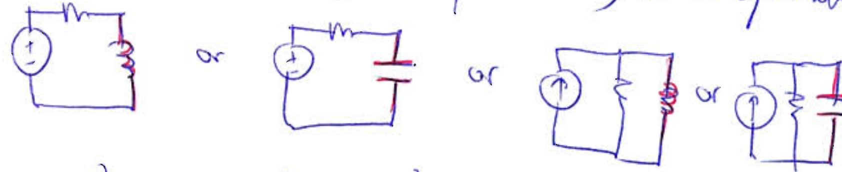
$f(0^+)$: usually by equilibrium & continuity
(if it's not a continuous variable, need to solve based on inductor currents and capacitor voltages found at equilibrium)

$f(\infty)$: equilibrium

τ: Often easiest by finding Thevenin resistance of the rest of the circuit.

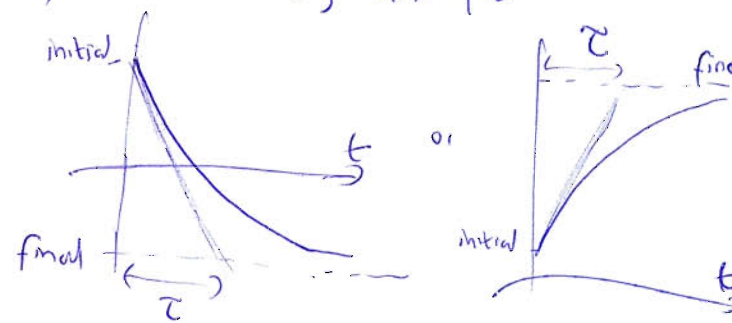
Summary of Thevenin/Norton method, and "curve fitting".

It may help to reduce the circuit (at the time period of interest) to its equivalent, then solve a simple case of

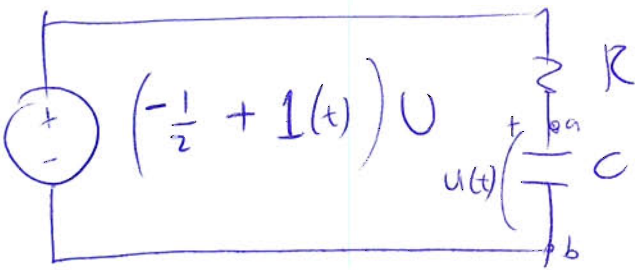


(Initial conditions can come from the original circuit's equilibrium).

If we just want the solution, for a first-order circuit we can find {initial value, final value, time constant} and fit a decaying exponential to these:



Let's take a simpler, <u>lab-relevant</u> case!



$$\begin{pmatrix} U = 5\,V \\ R = 10\,k\Omega \\ C = 100\,nF \end{pmatrix}$$

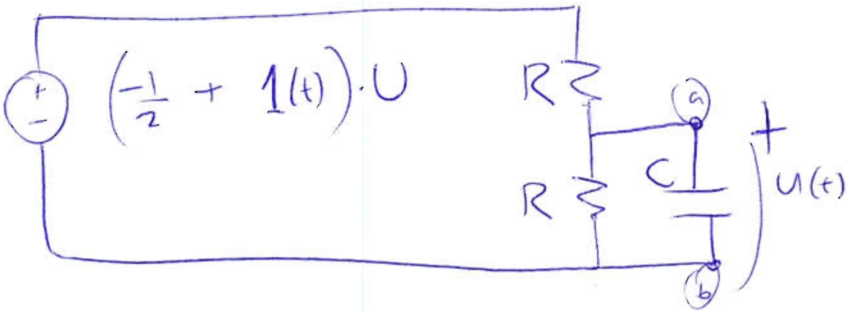
This is already in Thevenin form.

Determine the initial and final values and time-constant.

<u>Thereby</u>, find an expression for $u(t)$ at $t \geq 0$.

(see lab for more detail!)

Now consider a change (by adding a resistor parallel with the capacitor)



By considering how the Thevenin equivalent has changed at terminals a-b by adding the extra resistor, find the new function of $u(t)$.

(Notice the convenience of thinking in terms of the Thevenin source -- we more easily see how a change in the circuit will affect the time function.)