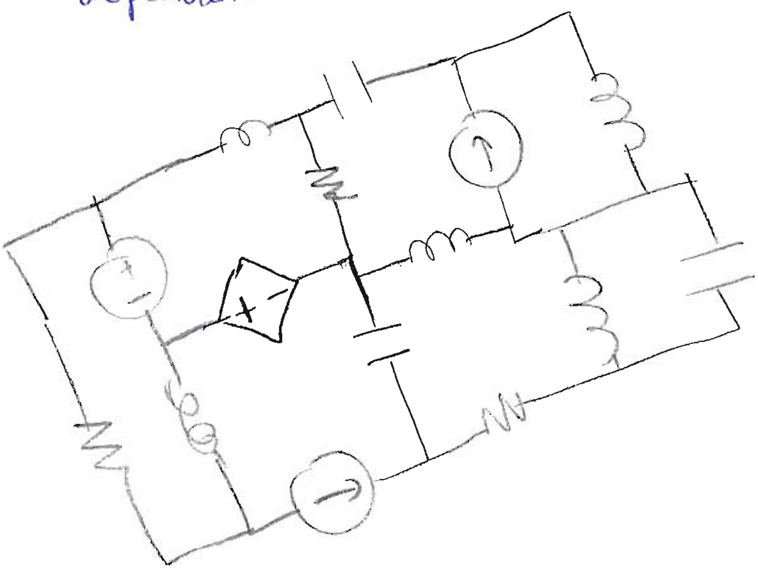
Introduction to AC analysis : phasors & impedances

(alternating current
vaxelström)

and conversion between phasors and time-functions

Consider a general* circuit of R, L, C, independent and dependent sources



*"general": not restricted to e.g. only one capacitor

- its full transient solution when changes happen may be very difficult ... ODEs of high order, many natural "modes" of oscillations

- the steadystate (equilibrium) with all sources constant (dc) is actually very easy: we did this already.

- now we consider another special situation where everything is SINUSOIDAL

"sinusoidal steady state" means that:

* the quantities of all independent sources vary sinusoidally in time, all at the same frequency

e.g.

$U_1(t) = U_{1peak} \cdot \sin(\omega t)$

$I(t) = I_{peak} \cdot \cos(\omega t + \varphi)$

$U_2(t) = U_{2peak} \cdot \sin(\omega t + \pi/2)$

same angular frequency

$\omega = 2\pi f$

radian/s

frequency eg. Hz

* the circuit is stable, and linear (we always assume that in this course), and has been left "long enough" with these sinusoidal sources so that the "natural response" in the circuit has damped away and all voltages and currents are just the "forced response" due to the sources

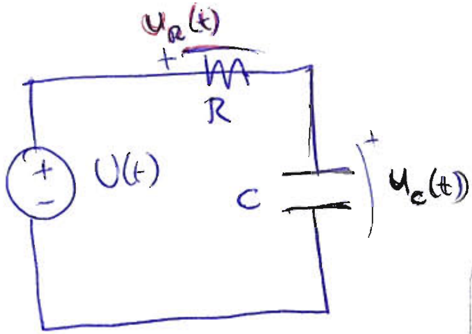
$\Rightarrow$ THEN ALL VOLTAGES AND CURRENTS ARE SINUSOIDAL

Sinusoidal steady state: "all quantities (u, i) have same shape as the sources – sinusoidal"

but they can differ in their size and timing
amplitude "phase"

This is not a general result for any periodic stimulus!
It's a special property of sinusoids.

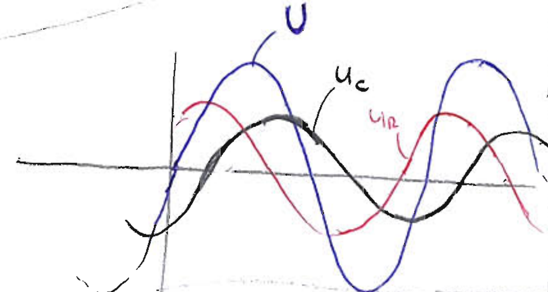
Remember the lab 3 (see the comments file):



U(t) sinusoidal

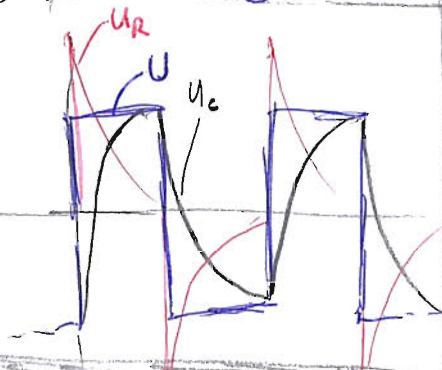
⇒ all quantities are sinusoidal

differ only in amplitude e phase



U(t) not sinusoidal, eg. square-wave

⇒ very different shapes are possible for u_c e u_R



Why is this "ac analysis", based on "sinusoidal steady state" useful?

$\Rightarrow$ Why devote so much time to it?

* solution is much easier than the full ODE transient solution for complex circuits

* many applications have sinusoidal (or approximately sinusoidal) stimulus (and have transients dying away relatively quickly)

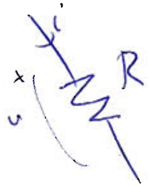
  e.g. the ac electric power system

  "carrier" of a radio station

* other periodic waveforms can be treated by their fourier series components — es. do ac analysis for each of the main (strongest) components

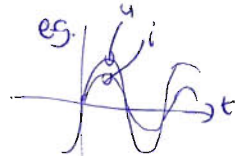
$\boxed{\text{So it's actually very important, in power and communications!}}$

Can we think of simple justifications of why the circuits with all sinusoidal stimulus will have only sinusoidal u e i everywhere?



if one quantity is sinusoidal, the other must be: $\frac{u}{i} = R$ at every point in time

eg.



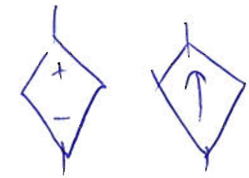
integral or derivative of sin or cos is another sin or cos function of same frequency

eg.

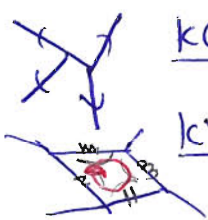
$$u = L\frac{di}{dt} \qquad u(t) = \omega L I \cos(\omega t)$$

$$i(t) = I\sin(\omega t) \qquad = \omega L I \sin(\omega t + \pi/2)$$

(example)



and dependent sources just "follow the controlling variable", so if that is a sinusoid the output will be a sinusoid.



KCL

KVL

adding or subtracting sinusoids of the same frequency gives another sinusoid of this frequency

this shows that it makes sense for sinusoidal quantities in all parts of the circuit being able to have a solution (it's no proof!)

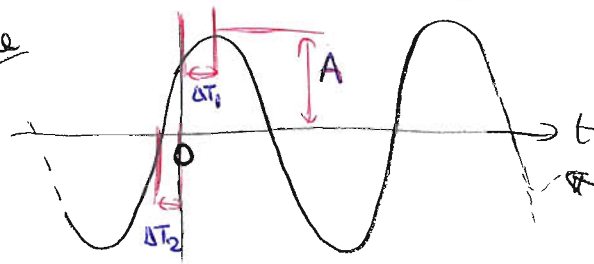
Basic principles: PHASOR fasvektor

By the assumptions for ac analysis,
any voltage or current will be sinusoidal, @ known angular frequency ω

In that case only two parameters are needed to specify (describe) its completely
$\Rightarrow$ amplitude and phase

angle added to whatever we choose as our reference function, to make this signal.

example



This can be written as:

$A\cos(\omega t - \varphi_1)$ where $\varphi_1 = \Delta T_1 \cdot \omega$

or $A\sin(\omega t + \varphi_2)$ where $\varphi_2 = \Delta T_2 \cdot \omega$

or $A\cos(\omega t - \frac{\pi}{2} + \varphi_2)$

etc.

phase, depending on whether the chosen function is sin or cos

Complex numbers have magnitude and angle.
They are well suited to describing these sinusoids.
A PHASOR is a complex number representing a sinusoidal quantity.

Basic principles: IMPEDANCE

Resistance is a ratio of $\frac{\text{voltage}}{\text{current}}$ ... hence $u = iR$ etc.

Impedance is a more general concept (but similar) for ac circuits.

Symbol usually Z

phasors

$$Z = \frac{u}{i}$$

may be complex as it is a ratio of phasors

for a resistor: simple .. $Z = R$. (purely real)

for an inductor: $Z = j\omega L$ (purely positive imaginary)

$\sqrt{-1}$

for a capacitor. $Z = \frac{1}{j\omega C} = \frac{-j}{\omega C}$ (purely negative imaginary)

What these impedances tell us is that:

- for a resistor, voltage and current are "in phase"
- for an inductor, voltage has phase 90° earlier than the current ($\pi/2$) ("leading")
  this is seen from the j which is a 90° phase shift in the complex plane.
- for a capacitor, voltage has phase 90° later than the current. ("lagging")
  this is seen from the $-j$

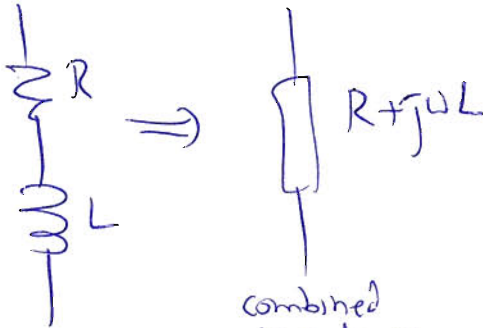
We can see this from earlier studies of C & L with sinusoids:

e.g.

$$u(t) = L\frac{di}{dt} = \omega L I \cos(\omega t)$$

$\cos(\omega t)$ 'leads' $\sin(\omega t)$ by 90°

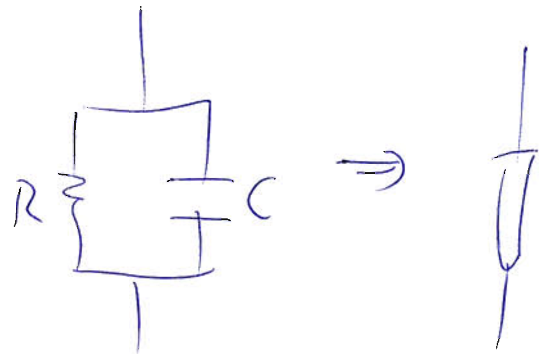
combining impedances ... just like resistances! but COMPLEX (numbers)



components

combined impedance

etc.



$$\frac{R\,\frac{1}{j\omega C}}{R+\frac{1}{j\omega C}}$$

$\Downarrow$

simplify to

$$\frac{R}{1+j\omega CR}$$

How to do ac analysis . . .

0 describe voltages and currents as phasors (known or unknown)
describe capacitors and inductors as impedances

1 Calculate as for dc circuit, but with phasors and impedances instead of purely real values of dc sources and resistance.

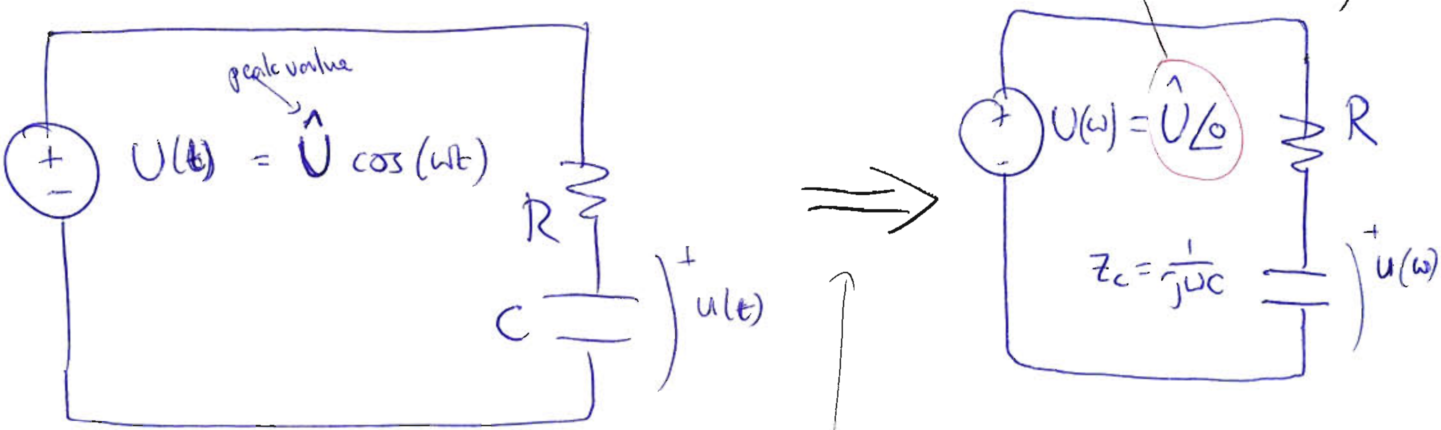
2 if necessary, convert the solution (a phasor) to the corresponding time-function

In many applied cases, we don't care about time functions at all, but only about amplitudes and relative phase -- then step 2 can be omitted.

In this topic, we mainly start with time functions, convert to phasors, calculate by ac analysis, then convert back to time functions.

A simple example:

complex number with magnitude of $\hat{U}$ and angle of 0 (in other words, purely real!)

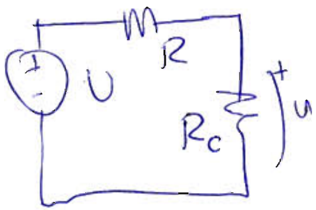


find $u(t)$

Convert from "time domain" to "frequency domain" (phasors)

Now treat the phasors and impedances as you would treat the source values and resistances in a dc analysis.

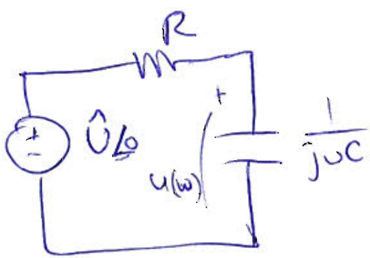
If it had been dc, it could have looked like this:



$\longrightarrow$ voltage division $\quad u = \dfrac{R_c}{R+R_c}\, U$

So in our case we have instead (complex)

Since the angle is 0 we can write just $\hat{U}$ as a short form of $\hat{U}\angle 0$



$$\longrightarrow \quad U(\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}\, \hat{U}\angle 0 = \frac{\hat{U}}{1 + j\omega CR}$$

This is a frequency domain solution: we have found a phasor u that describes the sought quantity.

We were asked to find $u(t)$, so we need to write the time function that this phasor describes:

Somehow we must convert $U(\omega) = \dfrac{\hat{U}}{1+j\omega CR}$ to $u(t) = A \cdot \cos(\omega t + \varphi)$

(frequency domain)

amplitude

phase

(time domain)

At the start we made a choice .... the source's time function of $U(t) = \hat{U}\cos(\omega t)$ was written as phasor $U(\omega) = \hat{U}$

real number equal to amplitude of the time function

* That means we chose that a cosine $\cos(\omega t)$ with no added phase corresponds to a phasor with zero phase (real number) — (this is called "cosine reference)

* And that a cosine with peak amplitude $\hat{U}$ corresponds to a phasor of magnitude $\hat{U}$. (peak reference)

Neither of these choices is a necessary one (we could choose that $\sin(\omega t)$ or $\cos(\omega t + \pi)$ gives a phasor of zero angle, or that the phasor's magnitude is scaled by $\frac{1}{2}$ or 3 etc). But we must use the same for $u(t) \to u(\omega)$ converting back into time, $U(\omega) \to u(t)$

So, with our choice of references,

$$u(t) = \begin{pmatrix} \text{MAGNITUDE} \\ \text{OF OUR PHASOR} \\ u(\omega) \end{pmatrix} \cdot \cos\left(\omega t + \begin{pmatrix} \text{ANGLE OF} \\ \text{OUR PHASOR} \\ u(\omega) \end{pmatrix}\right)$$

magnitude

$$u(t) = \left| \frac{\hat{U}\angle 0}{1+j\omega CR} \right| \cdot \cos\left(\omega t + \angle \frac{\hat{U}\angle 0}{1+j\omega CR}\right)$$

means "angle of"

Thus

$$\boxed{u(t) = \frac{\hat{U}}{\sqrt{1+(\omega CR)^2}} \cdot \cos\left(\omega t - \tan^{-1}(\omega CR)\right)}$$

see "chapter" file for complex number refresher!

How were these parts done?

magnitude of phasor

angle

$$\frac{V\angle 0}{1+j\omega CR} = \frac{A\angle\alpha}{B\angle\beta} = \frac{A e^{j\alpha}}{B e^{j\beta}} = \frac{A}{B} e^{j(\alpha-\beta)} = \frac{A}{B} \angle \alpha-\beta$$

$A = V$

$\alpha = 0$

$B = \sqrt{1^2 + \omega^2 C^2 R^2}$

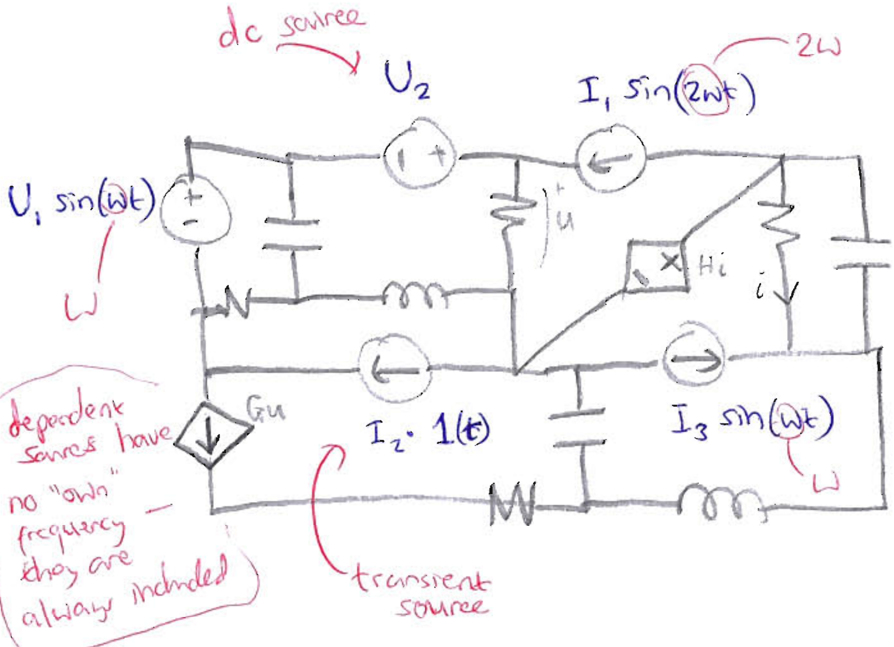
$\beta = \tan^{-1}\left(\frac{\omega CR}{1}\right)$

$$\frac{A}{B} = \frac{V}{\sqrt{1+(\omega CR)^2}}$$

$$\alpha - \beta = -\tan^{-1}\omega CR$$

FINAL POINT

If we have only some sources sinusoidal at a particular frequency — and others have dc ($f = 0$ Hz!) or other frequency, we can still do ac analysis, but using SUPERPOSITION to handle just the same-frequency sources at a time.



Could solve then superpose:

- ac analysis at ω (two sources active)
- ac analysis at 2ω (one source active)
- dc analysis (one source active)
- transient analysis (one source)