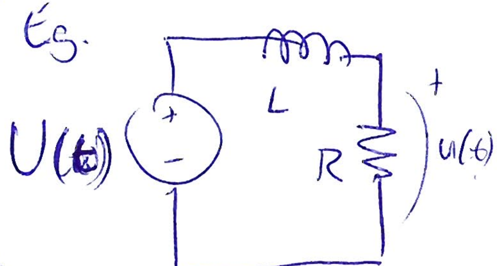
Filters and "frequency response" in general

We consider here, by ac analysis, the **frequency response** of circuits.
In particular, we look at "$\frac{\text{output}}{\text{input}}$" relations between phasor quantities.

$\Rightarrow$ huge, essential part of communications, "signals" generally, hi-fi, etc.
... important in some power applications too,
and frequency response is studied in many subjects outside electric circuits

E.g.

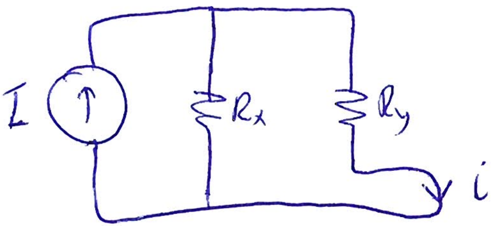


Let $L = 1\,H$ and $R = 1\,k\Omega$
What is the amplitude of $u(t)$ if:
i) $U(t) = (100\,V) \cdot \sin(2\pi \cdot 10\,kHz \cdot t)$
ii) $U(t) = (100\,V) \cdot \sin(2\pi \cdot 10\,Hz \cdot t)$

?

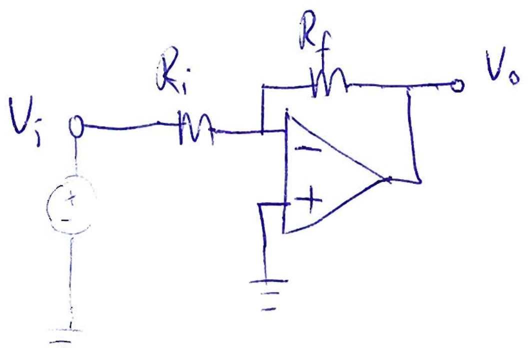
Input → Output relations $\left(\frac{\text{output}}{\text{input}}\right)$ of systems (e.g. circuits)

We have already seen some examples in dc: we called the ratio "gain".



$$i = \frac{R_x}{R_x+R_y} I \Longrightarrow \frac{i}{I} = \frac{R_x}{R_x+R_y}$$

constant, real, ratio of $\frac{\text{"output"}}{\text{"input"}}$



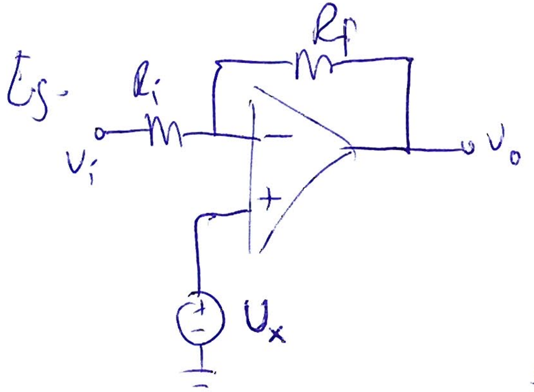
$$V_o = \frac{-R_f}{R_i} V_i \Longrightarrow \frac{V_o}{V_i} = \frac{-R_f}{R_i}$$

In ac circuits with C & L, the ratio may be $\left\{ \begin{array}{l} \text{complex \&} \\ \text{frequency-dependent} \end{array} \right.$

(A little reminder, for the sake of completeness / pedantry.)

We can't <u>always</u> express the actual values of $\frac{\text{output}}{\text{input}}$ as a ratio.

Eg.



$$V_o = U_x\left(1 + \frac{R_f}{R_i}\right) - \frac{R_f}{R_i} V_i$$

constant

proportional to input

* trying to find $\frac{V_o}{V_i}$ here would result in an expression that contains V_i (or V_o).

* instead, we define the "gain" as the ratio of <u>changes</u>: $\boxed{\frac{\Delta V_o}{\Delta V_i} = \frac{-R_f}{R_i}}$

or "$\frac{dV_i}{dV_o}$"

Many circuits used with ac have significant frequency response,
i.e. the output/input relation depends on frequency quite strongly.

Some circuits are **deliberately designed** for a particular response.
E.g. let low frequencies pass through, but make high frequencies be attenuated (minsked)
- for a bass loudspeaker

E.g. let just one narrow band of frequencies pass through easily
- for a radio "tuner"
- for "dumping" unwanted harmonic currents to ground in a power circuit

etc.

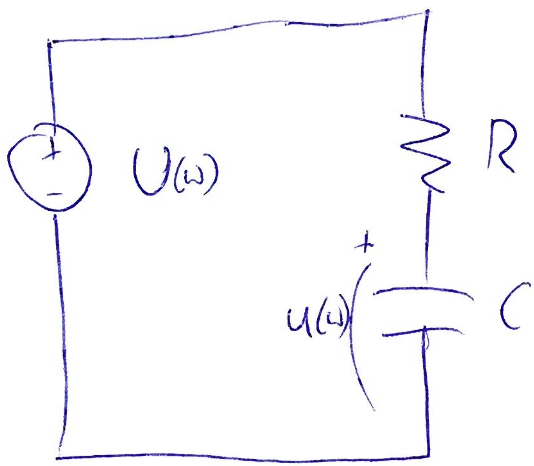
"pass" ← the usual technical words → "stop"

A circuit designed to allow some frequencies and block others is called a filter (obvious reason, associated with filtering e.g. particles from a liquid).

Now a genuine ac example.

(We'll work entirely with phasors – not translating from and to time functions.)

from the lab:

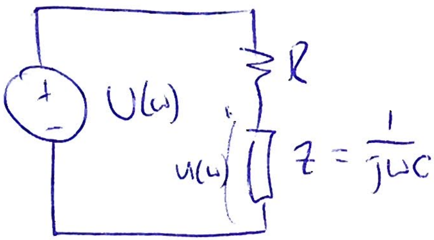


$$\frac{u(\omega)}{U(\omega)} = \underbrace{H(\omega)}_{} = \underset{\text{find me}}{?}$$

a "network function" – ratio of phasors of some out/in relation

(the name "transfer function" or "överföringsfunktion" is sometimes used to include this, or sometimes only for Laplace transform ratios)

Solution:



$$u(\omega) = \frac{\frac{1}{j\omega C}\, U(\omega)}{R + \frac{1}{j\omega C}} = \frac{U(\omega)}{1 + j\omega CR}$$

(This was by voltage division, as in the previous topic!)

Thus,

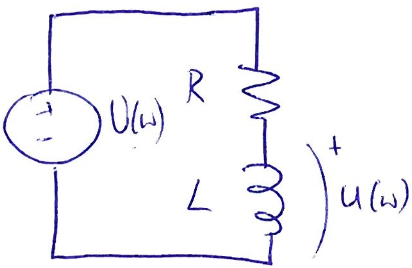
$$\boxed{\frac{u(\omega)}{U(\omega)} = H(\omega) = \frac{\left(\frac{U(\omega)}{1 + j\omega CR}\right)}{U(\omega)} = \frac{1}{1 + j\omega CR}}$$

note that this is frequency dependent

the magnitude and angle (phase) of $H(\omega)$ change with frequency,

(You would have seen this in the lab.)

Other examples:



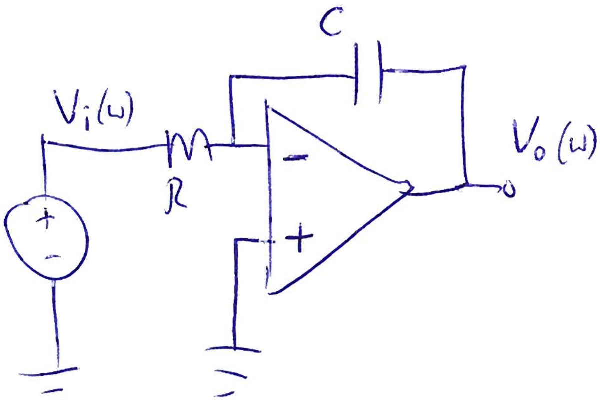
voltage division:

$$u(\omega) = \frac{j\omega L \, U(\omega)}{R + j\omega L}$$

$$\Rightarrow \boxed{\frac{u(\omega)}{U(\omega)} = \frac{j\omega L}{R + j\omega L}}$$

or, sometimes preferred:

$$\frac{u(\omega)}{U(\omega)} = \frac{j\omega L / R}{1 + j\omega L / R}$$



KCL at the "virtual earth" (0V opamp '−' input),

$$\frac{0 - V_o(\omega)}{1/j\omega C} + \frac{0 - V_i(\omega)}{R} = 0$$

$$\Rightarrow \boxed{\frac{V_o(\omega)}{V_i(\omega)} = \frac{-1}{j\omega CR}}$$

From these examples, we have seen the essential features of the network functions that will appear in this topic.

They can be written as a product/ratio of three types of term:

K constant

$(1 + j\omega/\omega_n)$

$j\omega/\omega_n$ ← constant

$$\text{Examples:} \quad H(\omega) = \frac{(1+j\omega/\omega_1)}{j\omega/\omega_2 \cdot (1+j\omega/\omega_3)}$$

$$\text{or} \quad H(\omega) = \frac{k}{1+j\omega/\omega_0}$$

Note: $\dfrac{j\omega/\omega_1}{j\omega/\omega_2}$ simplifies to $\dfrac{\omega_2}{\omega_1}$

but $\dfrac{1+j\omega/\omega_1}{1+j\omega/\omega_2}$ doesn't simplify unless $\omega_1 = \omega_2$

The previous examples of network functions could be written in this way as follows:

$$\frac{j\omega L/R}{1+j\omega L/R} = \frac{j\omega/\omega_0}{1+j\omega/\omega_0}$$

$$\left(\omega_0 = \frac{1}{\tau} = \frac{1}{L/R} = \frac{R}{L}\right)$$

$$\frac{-1}{j\omega CR} = \frac{K}{j\omega/\omega_1}$$

$$\left(k = -1, \quad \omega_1 = \frac{1}{\tau} = \frac{1}{CR}\right)$$

(Writing network functions based on these simple terms $\left\{ \begin{matrix} j\omega/\omega_0 \\ 1+j\omega/\omega_0 \end{matrix} \right\}$, with ω_0 real) is possible for our functions because of the types of fairly simple circuit we handle in this topic.

When there are <u>multiple</u> C ~~or~~ L components <u>and</u> they can <u>interact</u> we can easily get situations where some second-order terms are needed, (i.e. with an ω^2) or else our $1+j\omega/\omega_0$ terms would need complex ω_0. That <u>can</u> be handled, but we stick to the simplest cases for the course)

Soon we will see that it is convenient to keep our functions in the form like $(1+j\omega/\omega_1)(1+j\omega/\omega_2)$ and <u>not</u> to expand the parentheses. Then it is simpler to plot the $H(\omega)$ function.

Plotting / visualising / understanding $H(\omega)$.

This is important (in practice). Network functions can be complicated, so plotting really helps in comparing and assessing them.

$\boxed{\text{In general, } H(\omega) \text{ is complex.} \quad \text{(ie. 'usually')}}$

So how do we plot it as a function of frequency, unless able to do it in 3D?

Separate plots {for magnitude & phase
or real & imaginary parts} ?

The strong (norm) convention is:

plot magnitude $|H(\omega)|$ against frequency, log/log (y-scale, x-scale)

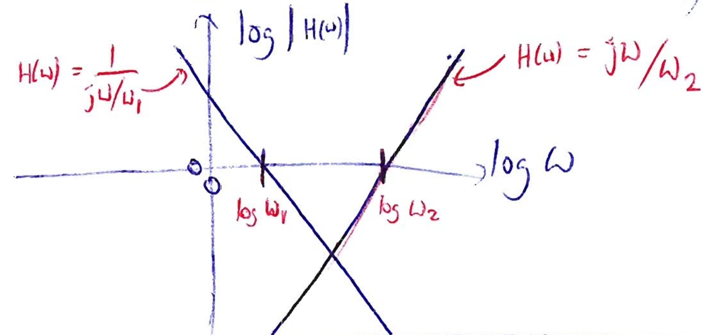
and possibly angle (phase) $\angle H(\omega)$ against frequency, lin/log

(Magnitude is often more important than phase ... it has a clear, locally defined meaning (measure with voltmeter/ammeter) whereas phase is relative to some other sinusoid. But phase sometimes is very important too.)

Why the logarithmic scale(s)?

- One usual reason: include a wide range of frequency (with same relative resolution [upplösning]).
- Also, convenient <u>shape</u> for magnitude:

  straight lines for terms with $j\omega/\omega_x$

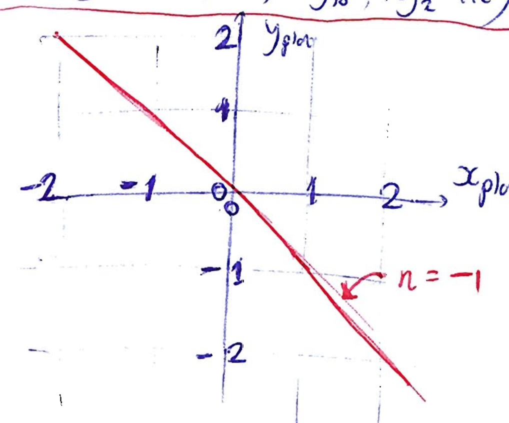


A general point:

if $y = x^n$, then plotting $\log y$ versus $\log x$ (with any choice: $\ln$, $\log_{10}$, $\log_2$ etc) gives a straight line with gradient n

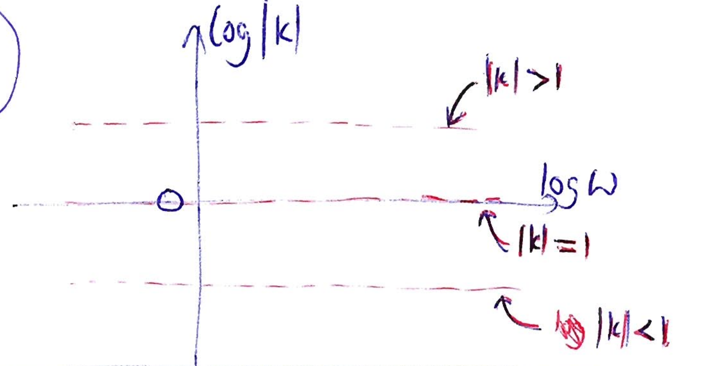
Why?

$$\left.\begin{aligned} x_{plot} &= \log x \\ y_{plot} &= \log x^n \end{aligned}\right\} \frac{y_{plot}}{x_{plot}} = \frac{\log x^n}{\log x} = n$$

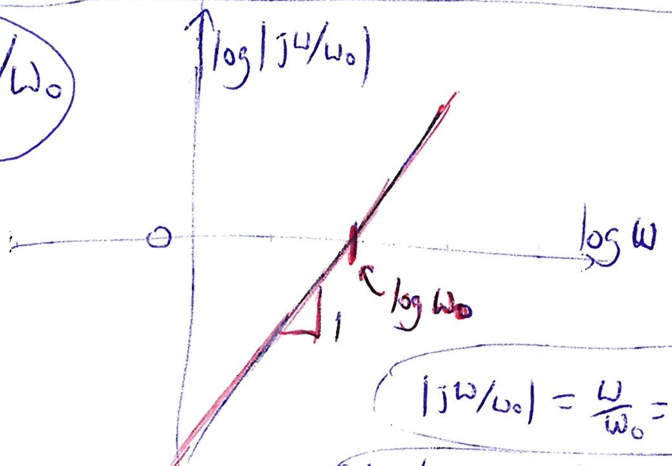


Now consider what the basic (first-order) terms look like when plotted in this way... We focus on amplitude.

K



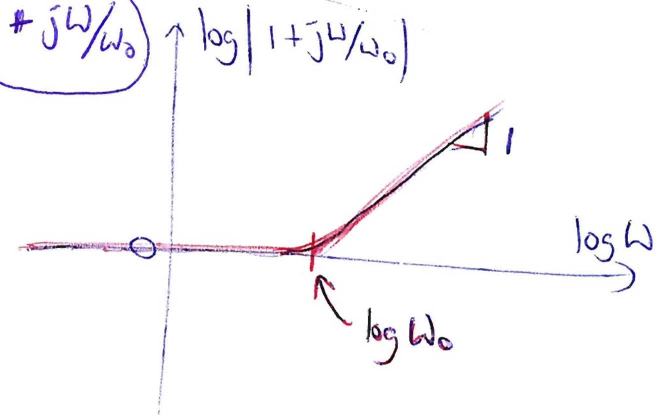
$j\omega/\omega_0$



$$\left|j\omega/\omega_0\right| = \frac{\omega}{\omega_0} = \frac{1}{\omega_0}\omega^1$$

$$\log\left(\frac{\omega}{\omega_0}\right) = 0 \text{ when } \omega = \omega_0$$

$1 + j\omega/\omega_0$



$$\omega \ll \omega_0 : \quad 1 + j\,\omega/\omega_0 \simeq 1$$

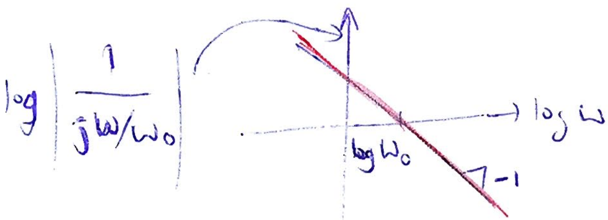
$$\omega \gg \omega_0 : \quad 1 + j\omega/\omega_0 \simeq j\omega/\omega_0$$

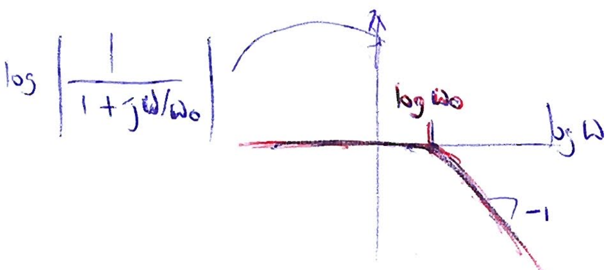
$$\omega = \omega_0 : \quad \left|1 + j\omega/\omega_0\right| = |1 + j| = \sqrt{2}$$

What if the terms are "underneath": $\frac{1}{j\omega/\omega_0}$ or $\frac{1}{1+j\omega/\omega_0}$?

Easy ... log-scale $\Rightarrow$ $\boxed{\log \frac{1}{x} = -\log x}$

So just "reflect in the x-axis".





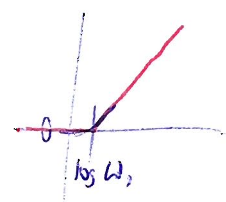
compare to the "on top" (non inverted) terms on the previous page.

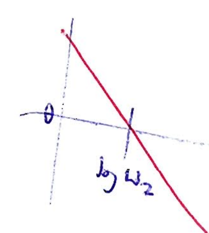
How about combining terms in a plot? Eg. $\dfrac{1+j\omega/\omega_1}{j\omega/\omega_2\,(1+j\omega/\omega_3)}$

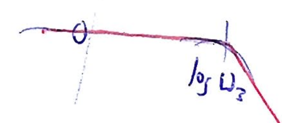
Well, ... $\log\dfrac{a}{bc} = \log a - \log b - \log c$

or $\log\dfrac{a}{bc} = \log(a) + \log\left(\dfrac{1}{b}\right) + \log\left(\dfrac{1}{c}\right)$

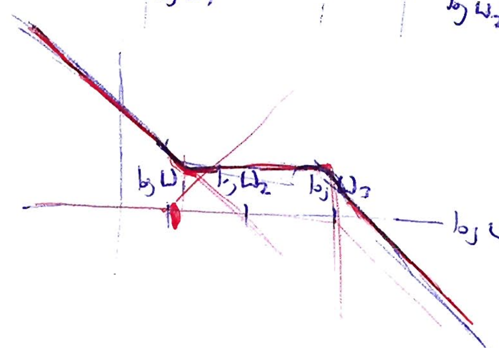
So just plot each term, eg $\left(1+j\omega/\omega_1\right)\cdot\left(\dfrac{1}{j\omega/\omega_2}\right)\cdot\left(\dfrac{1}{1+j\omega/\omega_3}\right)$







and add them:



In normal practice there is one more detail! (about magnitudes)

We don't just plot $\log(|H(\omega)|)$ versus $\log \omega$.

We plot $\underbrace{20 \log_{10}(|H(\omega)|)}_{\text{"decibel" scale.}}$ versus $\log_{10} f$!

What's decibel? A logarithmic power ratio. Short name: dB.

If $\log_{10}\left(\frac{P}{P_{ref}}\right) = 1$ this is "1 bel" i.e. P is $10\times$ the "reference" level

Decibels, defined as $10 \log_{10} \frac{P}{P_{ref}}$ are considered to give nicer numbers.

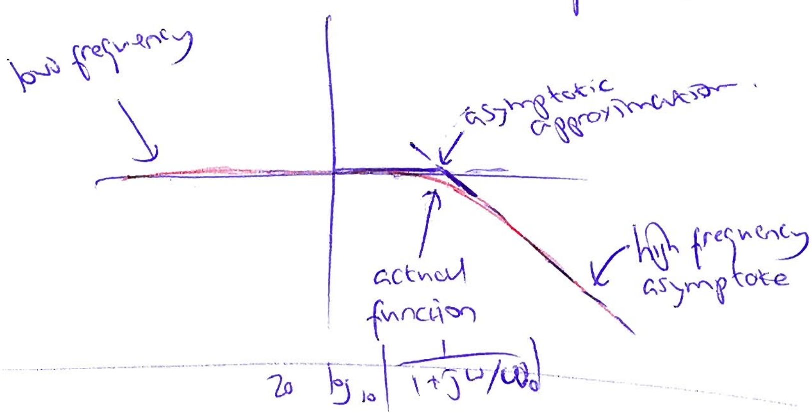
For network functions the input is the "reference": $10 \log_{10}\left(\frac{P_{out}}{P_{in}}\right)$

We are considering voltage or current, not power.

If a resistance is assumed, $P \propto u^2$ or i^2. Thus, $10 \log_{10} \frac{P}{P_{ref}} = 10 \log_{10}\left(\frac{u}{u_{ref}}\right)^2 = 20 \log_{10}|H|$

This type of plot: dB versus $\log_{10} f$, is called a Bode magnitude (or amplitude) plot.

It is classically drawn "asymptotically": instead of the curve when a $1+j\omega/\omega_0$ term changes its slope, the asymptotic plot has a sharp angle.



This introduces at most a $\simeq 3$dB error (at the point where $\omega = \omega_0$)

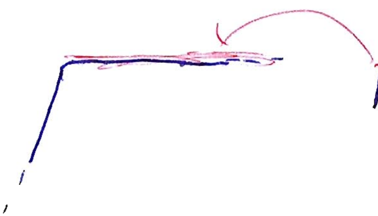
See "Chapter" for some nicely drawn examples of adding the dB plots of separate terms to give the complete magnitude plot of $H(\omega)$.

Lots of past exams have examples too, but often not showing the complete set of individual terms.

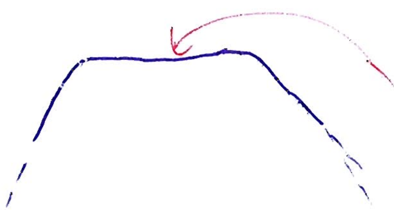
Some classic forms of (magnitude) frequency response.

(You do not need to memorise them by name!)

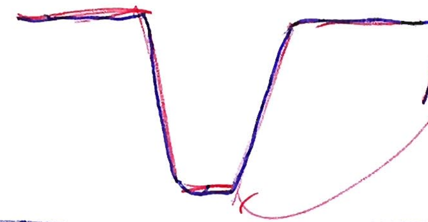
low-pass filter



high-pass filter



band-pass filter



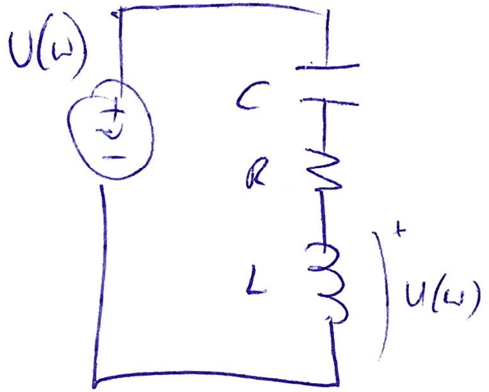
band-stop filter

These can all be made from our few simple terms, $j\omega/\omega_0$ or $1+j\omega/\omega_0$.

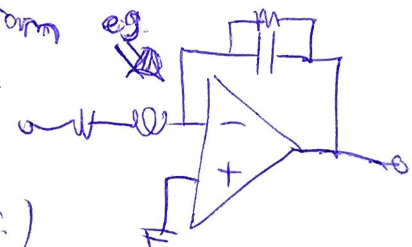
Many more complicated shapes can be made from combining multiple C & L (& R) components, with interaction ("second-order" terms).

("Digital filters" do the same sort of thing as an algorithm on samples of a signal!)

A brief, non-examined, look at a second-order filter.



← Here there is interaction between the capacitor and inductor – each affects the voltage the other sees (KVL).
(That's different from e.g.



where the inductor is not affected by the capacitor.)

$$\frac{u(\omega)}{U(\omega)} = \frac{j\omega L}{R + j\omega L + \frac{1}{j\omega C}}$$

$$= \frac{j^2\omega^2 LC}{j\omega CR + j^2\omega^2 LC + 1} = \frac{-\omega^2 LC}{j\omega CR + 1 - \omega^2 LC} = \frac{j\omega^2 LC}{\omega CR + j(\omega^2 LC - 1)}$$

(Not in the familiar first-order form.)

Example

(next page)

Example Make a Bode amplitude plot of $H(\omega) = \dfrac{j\omega/\omega_0}{1 + j\omega/\omega_1}$

(asymptotic plot of dB versus $\log \omega$ or $\log f$)

for

(a) $\omega_0 \ll \omega_1$

(b) $\omega_0 = \omega_1$

(c) $\omega_0 \gg \omega_1$

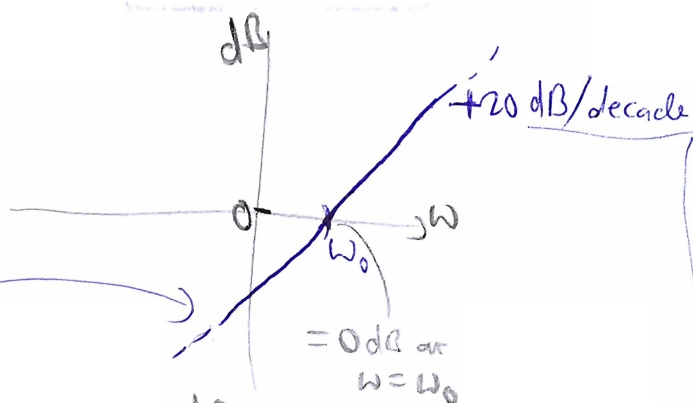
In dB,

$$|H(\omega)|_{dB} = \left|\frac{j\omega/\omega_0}{1 + j\omega/\omega_1}\right|_{dB} = 20\log_{10}\left(\frac{\omega/\omega_0}{|1 + j\omega/\omega_1|}\right) = 20\log_{10}\left(\frac{\omega/\omega_0}{\sqrt{1 + (\omega/\omega_1)^2}}\right)$$

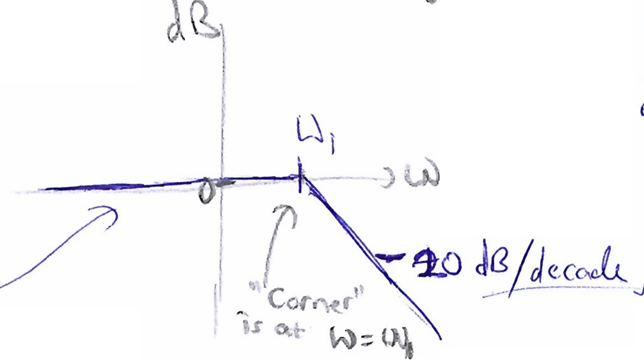
$$= 20\log_{10}(\omega/\omega_0) + 20\log_{10}\left(\frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_1}\right)^2}}\right)$$

So we can find the Bode amplitude plots of the separate multiplied terms in $H(\omega)$, then <u>add</u> their decibel values to find the total $|H(\omega)|$.

$$20\log_{10}\left(\left|j\omega/\omega_0\right|\right)$$



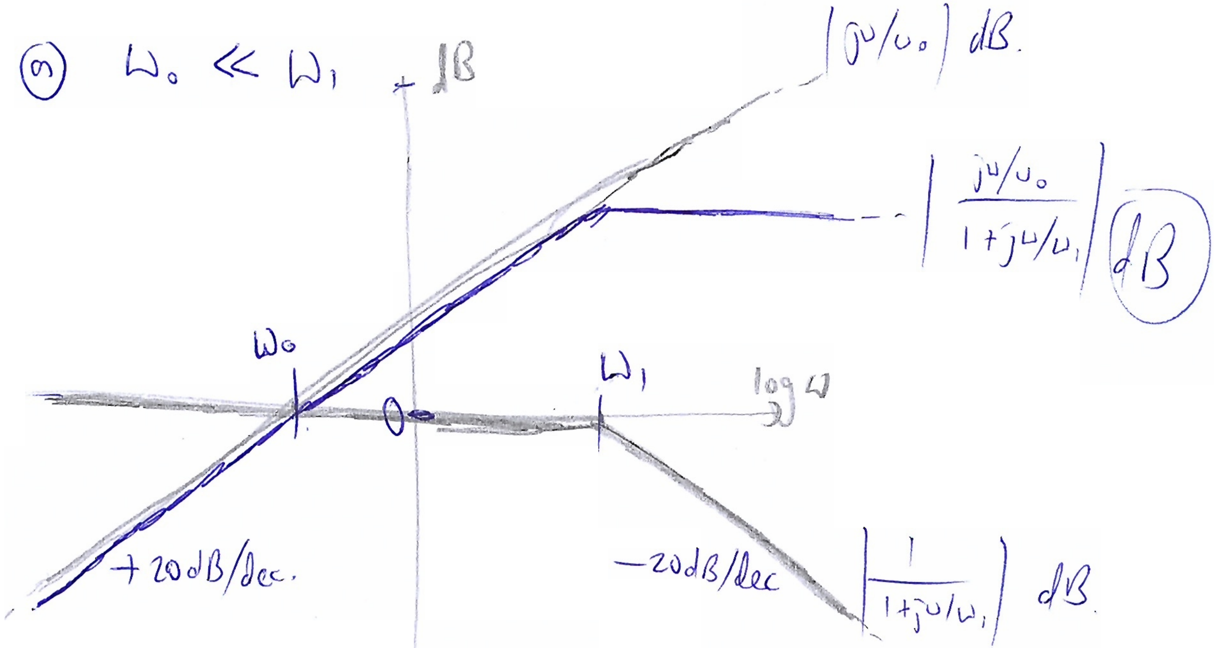
$$20\log_{10}\left(\left|\frac{1}{1+j\omega/\omega_1}\right|\right)$$



this means that the amplitude of the term increases or decreases 20 dB for every x10 change in frequency.

The resulting plot's shape depends a lot on relative values of ω_0, ω_1

(a) $\omega_0 \ll \omega_1$

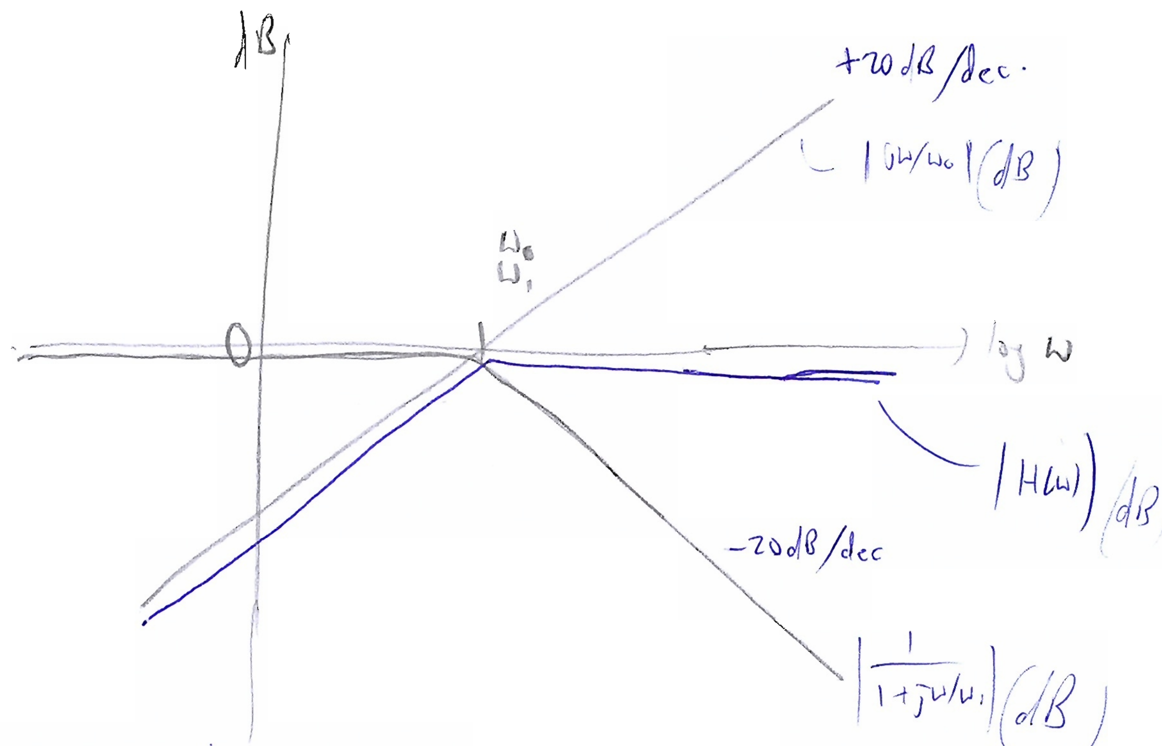


is the sum of the other two functions

in this region the total is the sum of 0dB (constant) and the +20dB/dec line passing through ω_0

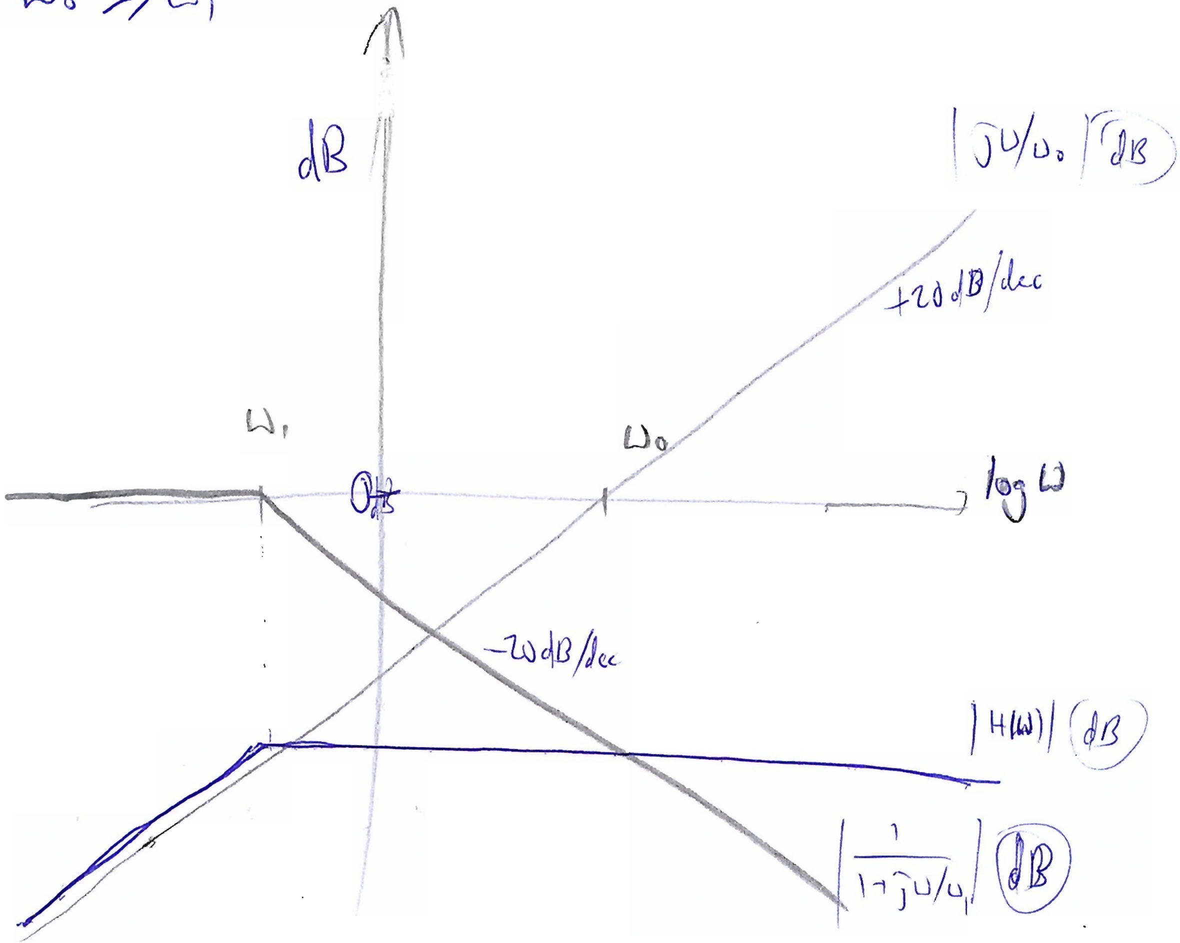
in this region, we add slopes of +20dB/dec & −20dB/dec, so the result is flat.

or (b) $\omega_0 = \omega_1$



The separate terms are the same as before, but shifted horizontally to fit $\omega_0 = \omega_1$. The result is a similar shape to the previous case, but shifted downwards.

(c) $\omega_0 \gg \omega_1$



Example

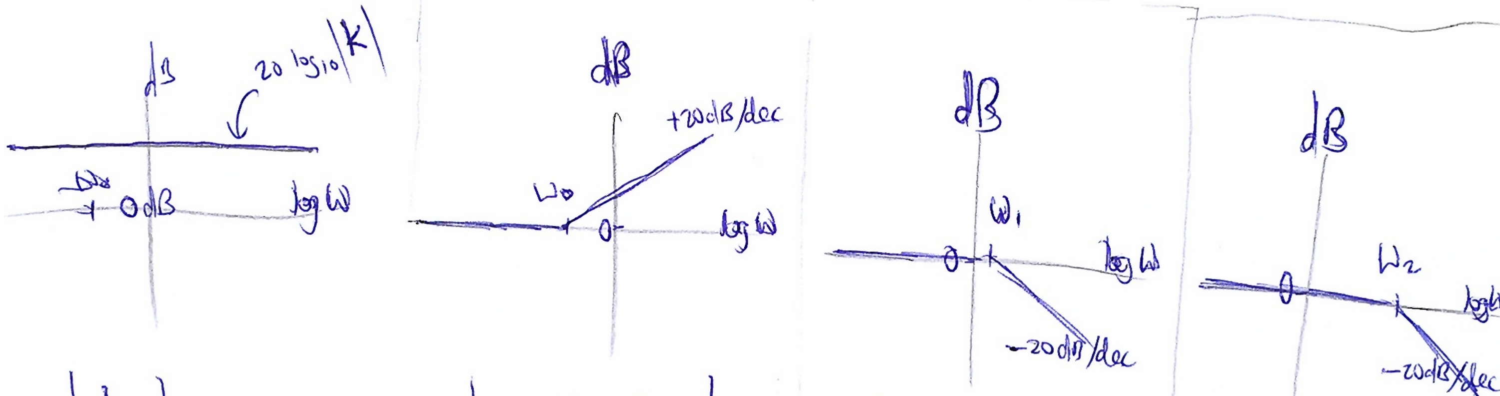
(next page)

More complex example, not so detailed explanation.

Make a Bode amplitude plot of $H(\omega) = \dfrac{K\,(1 + j\omega/\omega_0)}{(1+j\omega/\omega_1)(1+j\omega/\omega_2)}$

assuming $\omega_0 \ll \omega_1 \ll \omega_2$; and $|K| > 1$

There are 4 terms, which plotted separately are:



$|K|$ $\quad$ $|1 + j\omega/\omega_0|$ $\quad$ $\left|\dfrac{1}{1+j\omega/\omega_1}\right|$ $\quad$ $\left|\dfrac{1}{1+j\omega/\omega_2}\right|$

$(|K| > 1 \Rightarrow \text{dB} > 0\ ,\ \ |K| < 1 \Rightarrow \text{dB} < 0)$

Putting these together: