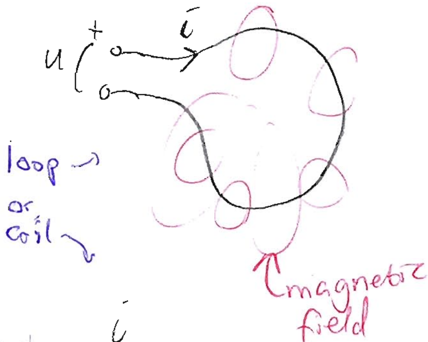
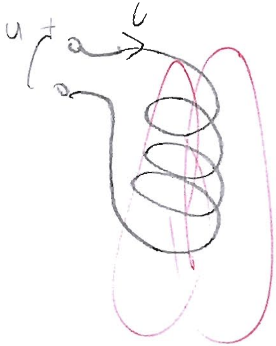
Mutual Inductance and Transformers

What we've seen already: inductors (separately)





$$\frac{u(t)}{di(t)/dt} = L \quad \text{general case (time domain)}$$

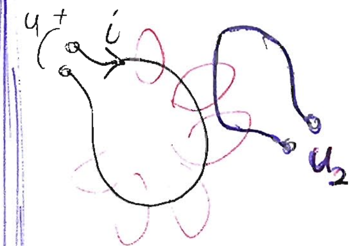
$$\frac{U(\omega)}{j\omega I(\omega)} = L \quad \text{phasor}$$

(self) inductance

current $\Rightarrow$ magnetic field

changing magnetic field $\Rightarrow$ voltage.

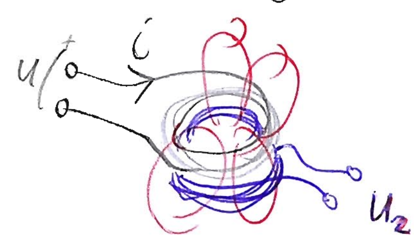
What we now study:



Relation of u e i

between different "coils"

that have some magnetic linking.

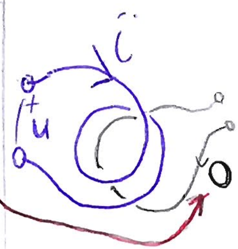


MUTUAL INDUCTANCE

consider two "coils"

could even be a single loop

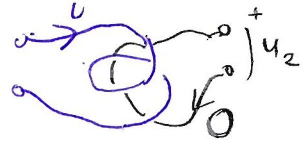
if one is "open circuit", it has no current, so does not make magnetic field, so does not affect the other ... then the other is a simple inductor.



$$\frac{u(t)}{di(t)/dt} = L$$

(as before)

-- but, when current flows in one coil it can induce voltage in the other due to the coupled (linked) magnetic field



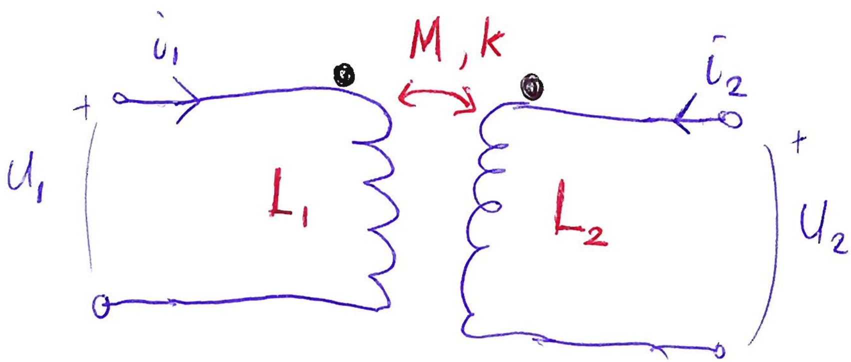
eg. $\frac{u_2(t)}{di(t)/dt} =$ "M"

mutual inductance (omsesidig induktans)

similar to L, but relates one coil's current to another's voltage

(sometimes called L_{12} between inductors 1 & 2)

Definitions for two coupled inductors:



L_1, L_2: self inductances of the individual coils alone.

M: mutual inductance between the coils

k: coupling coefficient (or factor)

These aren't independent!

$$M = k\sqrt{L_1 L_2}$$

$$u_1 = j\omega i_1 L_1 + j\omega i_2 M$$

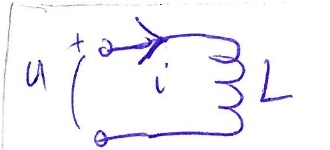
$$u_2 = j\omega i_2 L_2 + j\omega i_1 M$$

same mutual inductance for both directions $1\to2$, $2\to1$

the part we have used earlier with simple inductors

the part due to the "other" currents magnetic field

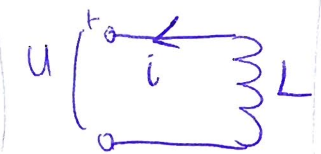
● ● the "dots" show the relative directions of the two coils (more on this later) for a plain inductor, relative direction of u & i definitions determines the sign in the component's equation:



$$u = +j\omega i L$$

passive

"convention"

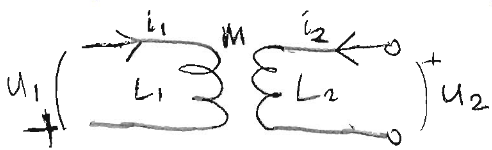


$$u = -j\omega i L$$

active

The same is true for each separate (coil) in coupled inductors!

inductor



$$u_1 = -j\omega L_1 i_1 \pm j\omega M i_2$$

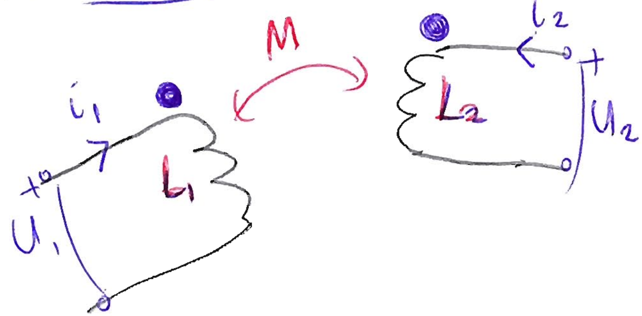
$$u_2 = +j\omega L_2 i_2 \pm j\omega M i_1$$

no dots to show the relative direction

Can see from the "convention"

? What are the relative directions of the "coils"

The dots.

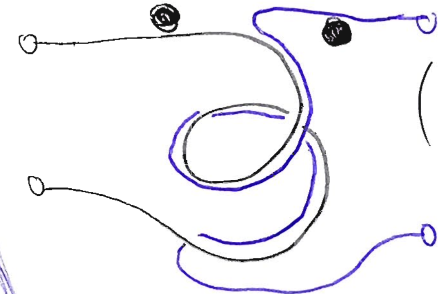


If current i_1 going into the dot on L_1 makes a positive term in the voltage u_1 equation, then current i_2 into the dot on L_2 also makes a positive term in this equation.

$$u_1 = + j\omega L_1 i_1 \quad + j\omega M i_2$$

because of directions of i_1, u_1 (passive)

same sign because i_1 & i_2 have same direction relative to dots



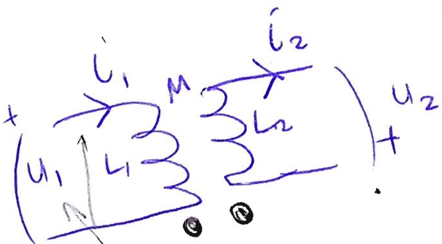
(Physical meaning of the dots)

In this simple geometry the dots mean "these ends of the inductors are at the same end of the magnetic path

--- currents into the dots travel the same direction

--- induced voltages have the same direction relative to the dots.

examples with dots



passive convention on L_1

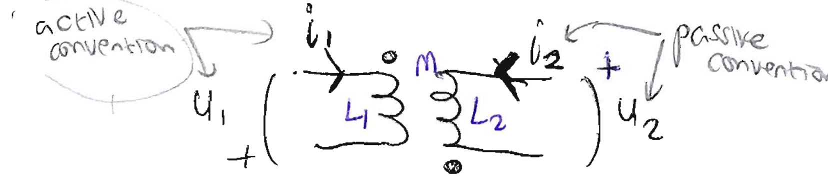
$$u_1 = +j\omega L_1 i_1 - j\omega M i_2$$

i_2 goes "in" to the dot on L_2 but i_1 goes "out" of the dot on L_1

Sign of i_2 term is opposite to sign of i_1 term

passive convention on L_2

$$u_2 = +j\omega L_2 i_2 - j\omega M i_1$$

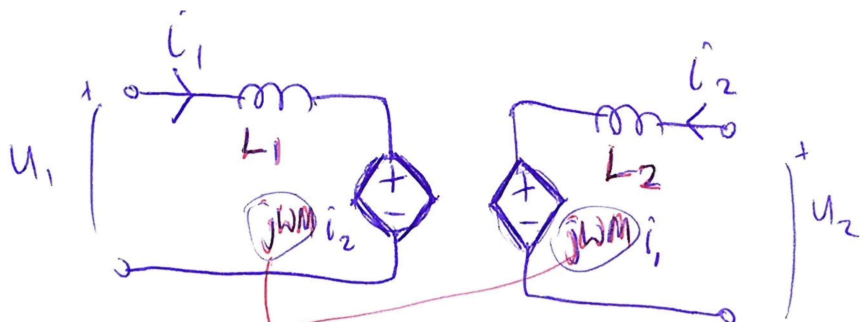


$$u_1 = -j\omega L_1 i_1 + j\omega M i_2$$
$$u_2 = +j\omega L_2 i_2 - j\omega M i_1$$

Currents go opposite, relative to the dots, so the signs of the i_2 terms are opposite to those of the i_1 terms

Another way to think of mutual inductance (in terms of already known components)

from the equations:
$$\begin{cases} u_1 = j\omega L_1 i_1 + j\omega M i_2 \\ u_2 = j\omega L_2 i_2 + j\omega M i_1 \end{cases}$$



dependent sources value

Nothing special ... no great insight ... just another way to think of it, that some people might like.

Calculations with mutual inductances.

coupling

If you only know (k), L_1, L_2 then find $M = k\sqrt{L_1 L_2}$

As usual: * calmly write all your knowledge of the circuit.

— define any unknown u or i that are needed

— write equation for mutual inductance: $\begin{cases} u_1 = j\omega L_1 i_1 + j\omega M i_2 \\ u_2 = \text{etc} \end{cases}$

careful about signs / dots

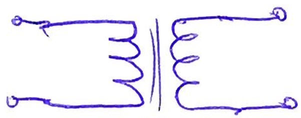
— write other circuit equations (parts outside the mutual inductance)

* then try to solve (& check dimensions!)

KVL is often useful, taken at each side of the ~~mutual~~ mutual inductors.

not always

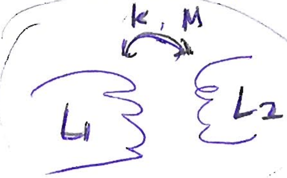
TRANSFORMERS — the ideal transformer



"ideal"

often don't bother to write this

Ideal transformer can be expressed as mutual inductors & where

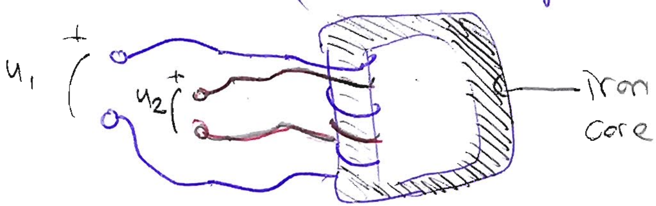


$$\{L_1, L_2\} \rightarrow \infty$$

$$k = 1$$

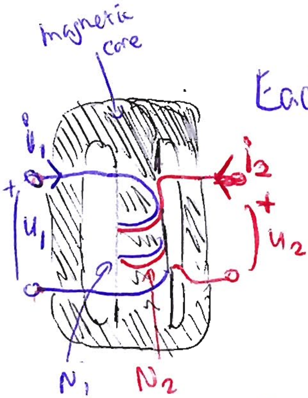
$$\therefore M \rightarrow \infty$$

Practically:
- good "magnetic circuit" (iron) — thick, short, high μ_0,
- linking both inductors closely
- (not much path for flux to "leak" (link one winding but not the other))



describe the number of turns on each coil as N_1 & N_2

Now, with N_1 "turns" around the magnetic core on one side ("primary") and N_2 on the other ("secondary"), and with each turn linking around all the magnetic field (perfect coupling), ...



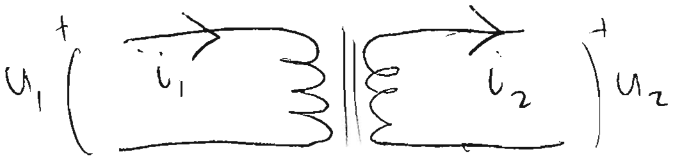
Each <u>turn</u> has the same voltage induced (same magnetic field)

$$\Rightarrow \frac{u_1}{N_1} = \frac{u_2}{N_2} \Rightarrow \boxed{\frac{u_2}{u_1} = \frac{N_2}{N_1}}$$

Currents must practically cancel each other, i.e. total current around the magnetic core $\simeq 0$, else the very permeable material would carry big magnetic field leading to <u>big</u> voltage that <u>forces</u> the currents to reach balance.

$$i_1 N_1 + i_2 N_2 \simeq 0 \Rightarrow \boxed{\frac{i_2}{i_1} = -\frac{N_1}{N_2}}$$

Commonly we define one current "out" to avoid the negative:



$N_1 : N_2$

or define $1 : r$ where $r = \frac{N_2}{N_1}$

ratio

$$\boxed{\frac{u_2}{u_1} = \frac{N_2}{N_1} \qquad \frac{i_2}{i_1} = \frac{N_1}{N_2}}$$

opposite "scaling" factors

$u \uparrow \Rightarrow i \downarrow$

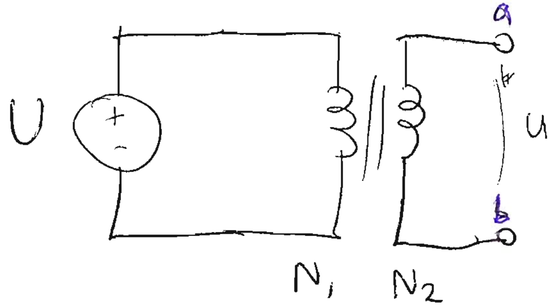
(same power at both sides)

Notice: ideal transformer is defined purely by its "turns ratio" eg N_2/N_1,

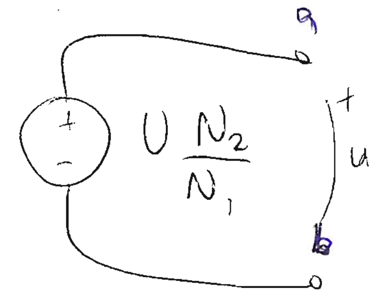
$\Rightarrow$ a single number instead of three (L_1, L_2, M).

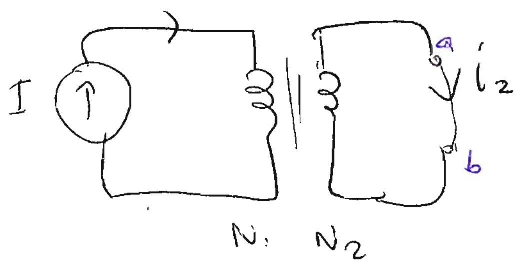
If nothing is connected to one side (open) no current flows in the other.

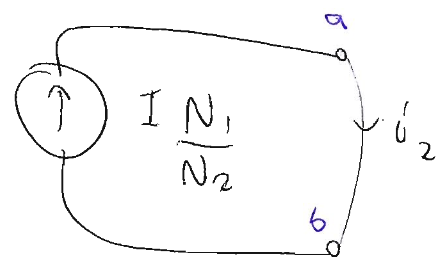
"Transferring across" the transformer.



seen at the terminals a-b this is the same as

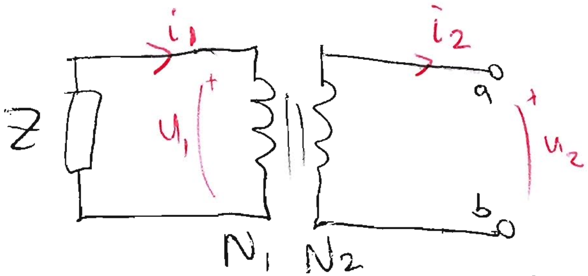




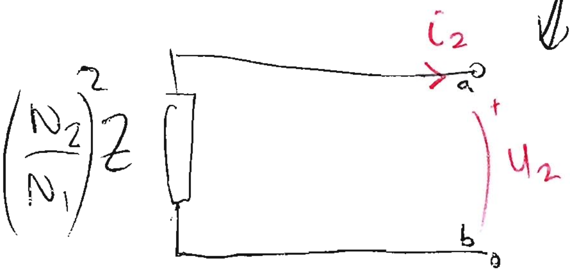


Transferring an impedance

First, a derivation based on the same circuits as used with the sources (then we get some negative signs because of having relative directions of u e i in the "active convention").



equivalent at a-b



$$i_2 = \frac{N_1}{N_2} i_1$$

$$U_1 = \frac{N_1}{N_2} U_2$$

transformer equations

$$i_1 = \frac{-u_1}{Z}$$

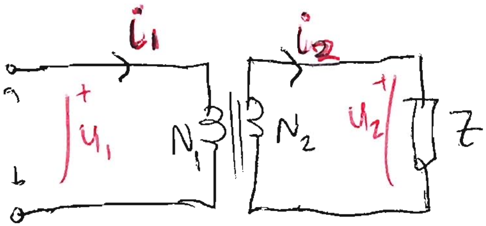
Ohm's law (active convention)

Putting these together,

$$i_2 = \frac{N_1}{N_2} \cdot \frac{N_1}{N_2} \cdot \frac{-U_2}{Z}$$

So $\boxed{\frac{-U_2}{i_2} = \left(\frac{N_2}{N_1}\right)^2 Z}$

alternatively, we can be more conventional, putting the impedance on the right-hand side (classic "load", with source on the left).



$$\left.\begin{aligned} i_1 &= \frac{N_2}{N_1} i_2 \\ u_2 &= \frac{N_2}{N_1} u_1 \end{aligned}\right\} \text{transformer}$$

$$\left. i_2 = \frac{u_2}{Z} \right\} \text{Ohm's law}$$

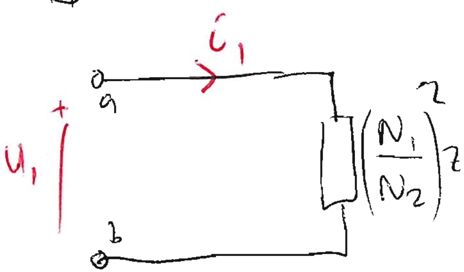
$$i_1 = \frac{N_2}{N_1} \frac{N_2}{N_1} \frac{u_1}{Z}$$

so

$$\frac{u_1}{i_1} = \left(\frac{N_1}{N_2}\right)^2 Z$$

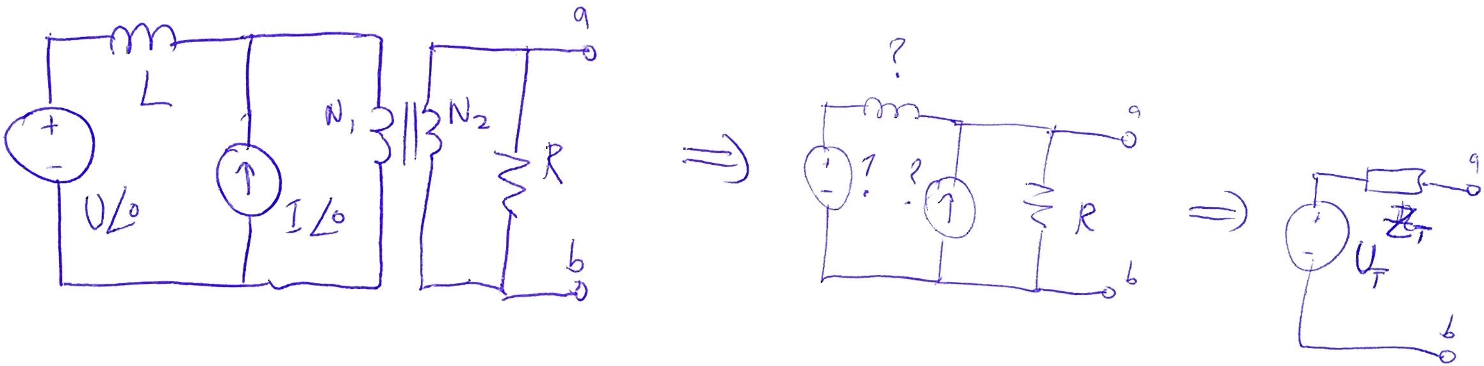
the impedance that behaves the same as the original impedance did when on the N_2 side of an $N_1 : N_2$ transformer.

equivalent when seen at the terminals a-b.

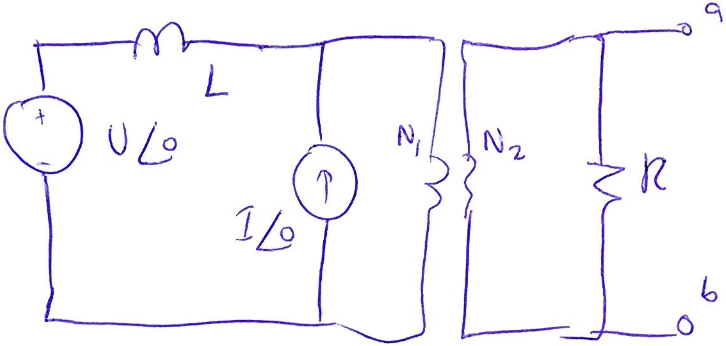


In some cases (typically when only interested in a solution at one side) it is helpful to "remove" the transformer after converting all components on the other side according to the turns ratio.

Eg. find Thevenin a-b.



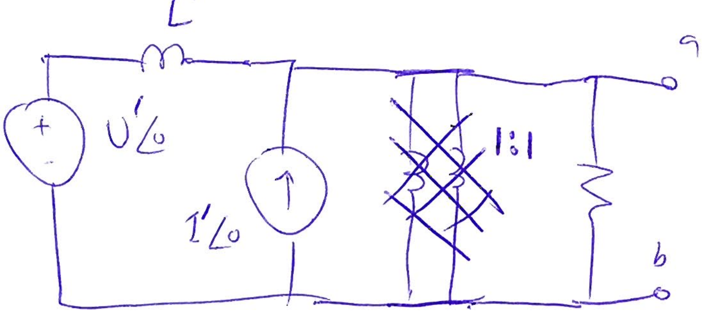
See next.

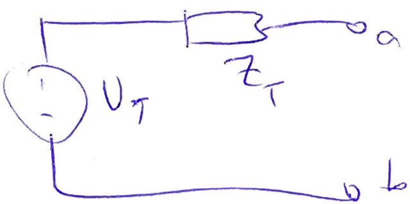


now transfer U, I, L to the other side:

$$U' = \frac{N_2}{N_1} U \qquad I' = \frac{N_1}{N_2} I$$

$$j\omega L' = j\omega \left(\frac{N_2}{N_1}\right)^2 L$$





$$U_{ab(oc)} = U_T$$

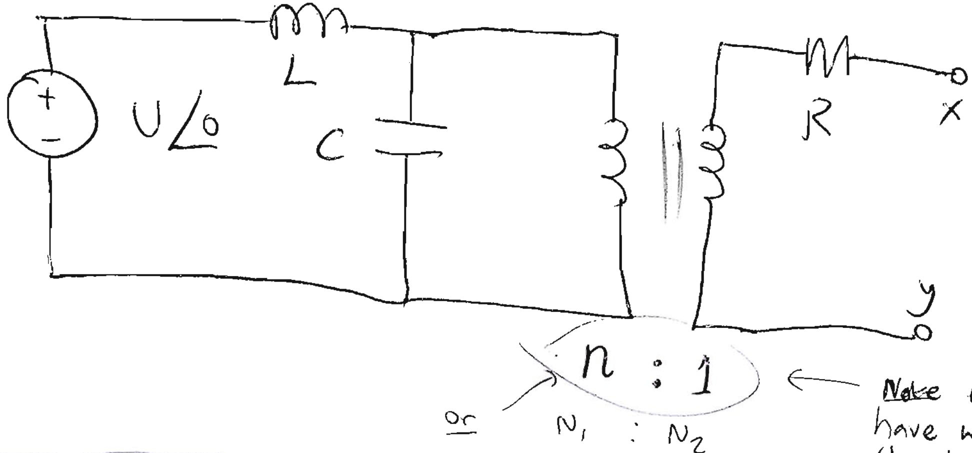
KCL: $\frac{u_{oc}}{R} + \frac{u_{oc} - U'}{j\omega L'} - I = 0$

$$\Rightarrow u_{oc} = \frac{(U' + j\omega L' I')\cancel{R}}{1 + j\omega L'/R}$$

$$\therefore U_T = \frac{\frac{N_2}{N_1} U + j\omega L I \frac{N_2^{\cancel{2}}}{N_1^{\cancel{2}}} \cancel{\frac{N_1}{N_2}}}{1 + j\omega L \left(\frac{N_2}{N_1}\right)^2 / R}$$

$$Z_T = R \,\|\, j\omega L' = \frac{j\omega \left(\frac{N_2}{N_1}\right)^2 L R}{R + j\omega \left(\frac{N_2}{N_1}\right)^2 L}$$

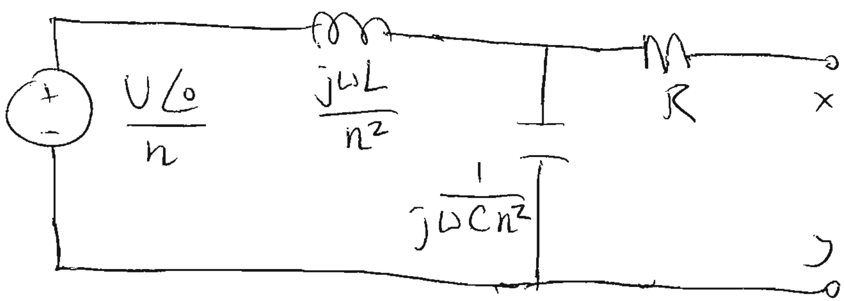
Example of transformer solution with "transferring" components.



find Norton equivalent at x–y.

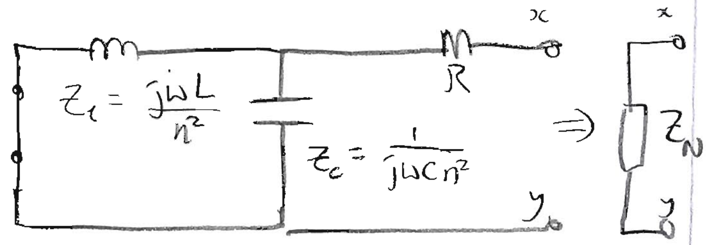
Note that we could equally well have written 1 : n, in which case the later solutions would have ×n instead of ÷n

We care about what is seen at x-y. So let's transfer everything to the right-hand side of the transformer.:



(Then solve in the usual way.)

Norton impedance (source set to zero)

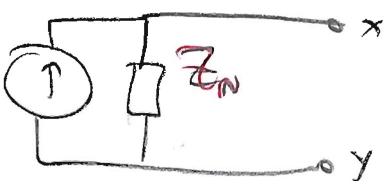


$$Z_N = R + \frac{Z_L Z_c}{Z_L + Z_c} = R + \frac{\left(\frac{j\omega L}{n^2}\right)\left(\frac{1}{j\omega C n^2}\right)}{\frac{j\omega L}{n^2} + \frac{1}{j\omega C n^2}}$$

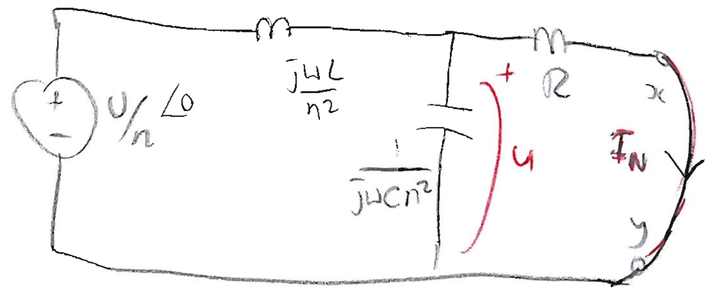
$$\boxed{Z_N = R + j\frac{1}{n^2\left(\frac{1}{\omega L} - \omega C\right)}}$$

Note the complexity came from the ac analysis, not from the transformer

ANSWER



Norton current (short-circuit, x-y)



KCL above capacitor:

$$\frac{u - \frac{U\angle 0}{n}}{j\omega L/n^2} + \frac{u}{1/(j\omega C n^2)} + \frac{u}{R} = 0$$

$$u\left(\frac{n^2}{j\omega L} + j\omega C n^2 + \frac{1}{R}\right) = \frac{U\angle 0 \; n^{\cancel{2}}}{j\omega L \cancel{n}}$$

$$u = \frac{U\angle 0 \; n}{j\omega L\left(\frac{n^2}{j\omega L} + j\omega C n^2 + \frac{1}{R}\right)}$$

$$\boxed{I_N = \frac{u}{R} = \frac{U\angle 0}{nR(1-\omega^2 LC) + j\omega L/n}}$$