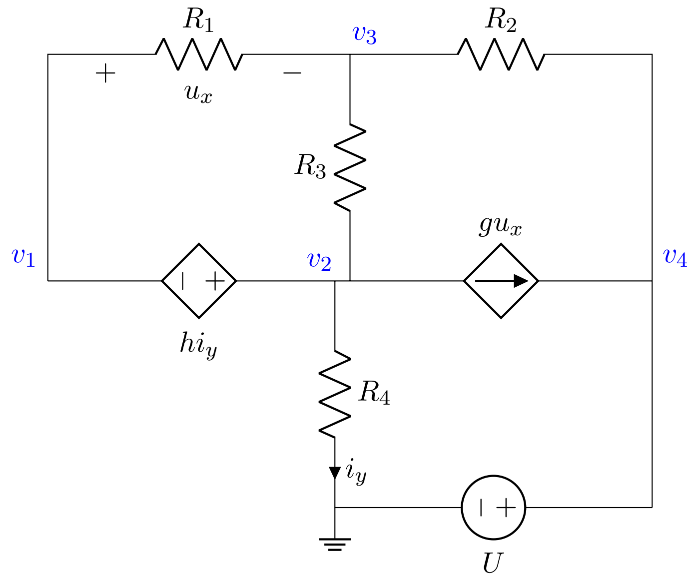
KTH, Electric Circuit Analysis, EI1120 VT-2021

Solutions for Tutorial 03 (Nodal Analysis)

These were in fact old exam questions, from
2014-02_EM_ks1.pdf Question 3 and 2016-03_EM_tenta.pdf Question 2

1.



We'll number the nodes 0 (earth), 1 (left), 2 (centre), 3 (top), 4 (right), as shown above. You might have made different choices.

Extended nodal analysis without simplifications

Define the unknown currents in voltage sources U and hi_y as i_α and i_β respectively, into the sources' + poles (passive).

KCL at all nodes except earth:

$$\text{KCL}(1): \quad 0 = \frac{v_1 - v_3}{R_1} - i_\beta \tag{1}$$

$$\text{KCL}(2): \quad 0 = \frac{v_2 - v_0}{R_4} + i_\beta + gu_x + \frac{v_2 - v_3}{R_3} \tag{2}$$

$$\text{KCL}(3): \quad 0 = \frac{v_3 - v_1}{R_1} + \frac{v_3 - v_2}{R_3} + \frac{v_3 - v_4}{R_2} \tag{3}$$

$$\text{KCL}(4): \quad 0 = \frac{v_4 - v_3}{R_2} - gu_x + i_\alpha \tag{4}$$

Now there are 8 unknowns ($v_0 \ldots v_4, u_x, i_\alpha, i_\beta$), and 4 equations. If we use the information that one node (v_0) has been defined as a zero-reference (earth), there's a 5th equation:

$$v_0 = 0. \tag{5}$$

Adding KCL at the earth node does *not* provide a useful equation: with N nodes the Nth node's KCL is just a linear combination of the KCL equations, so it provides no extra information. Instead, use the information given by the voltage sources:

$$\begin{aligned} v_2 - v_1 &= h i_y && (6) \\ v_4 - v_0 = v_4 &= U && (7) \end{aligned}$$

Now there are 7 equations, but 9 unknowns because i_y has been introduced. So define the controlling variables of the dependent sources in terms of existing variables – then there are 9 equations and 9 unknowns.

$$u_x = v_1 - v_3 \tag{8}$$

$$i_y = \frac{v_2}{R_4} \tag{9}$$

The systematic way in which this was done is important! There are plenty of ways to write a sufficient set of equations, but we cannot just be confident that "n unknowns, n equations, therefore it's all ok" is true. The above method of handling $N-1$ nodes, then earth potential and the information given by voltage-sources, then defining controlling variables in terms of known variables, is one way to develop linearly independent equations.

Supernode and Simplifications method (fewer equations, more thinking)

Nodes 0 and 4 become a supernode (an earth supernode); we choose to always use U instead of v_4 in the equations (and 0 instead of v_0).

Nodes 1 and 2 become another supernode; we choose to define unknown potential v_2, and always write $(1 - h/R_4)v_2$ instead of v_1.

Node 3 is a further node, with unkown potential v_3.

We define the dependent current-source's current in terms of our chosen node potentials as $gu_x = g((1 - h/R_4)v_2 - v_3)$.

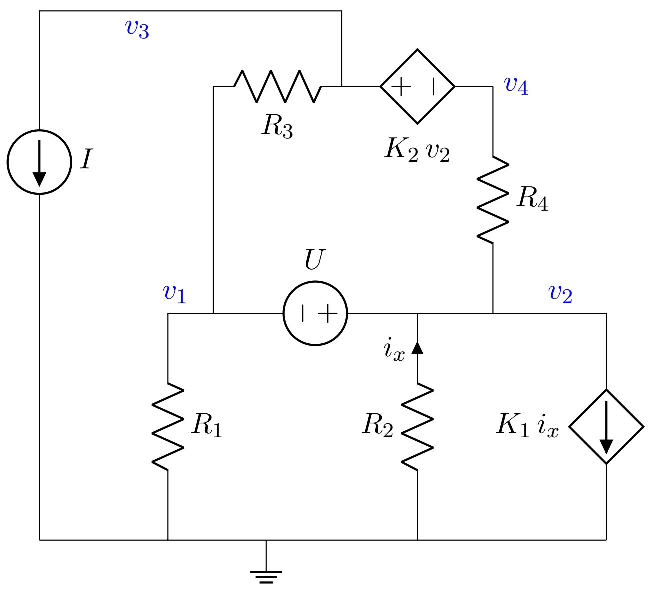
There are now only two unkown variables: v_2 and v_3.
Writing KCL at the two non-earth nodes/supernodes,

$$\begin{aligned} \mathrm{KCL}(1\&2): \quad 0 &= \frac{v_2}{R_4} + \frac{v_2 - v_3}{R_3} + \frac{(1 - h/R_4)v_2 - v_3}{R_1} + g\left(\left(1 - \frac{h}{R_4}\right)v_2 - v_3\right) && (1) \\ \mathrm{KCL}(3): \quad 0 &= \frac{v_3 - (1 - h/R_4)v_2}{R_1} + \frac{v_3 - v_2}{R_3} + \frac{v_3 - U}{R_2} && (2) \end{aligned}$$

To do as the question required, one should also write as equations the earlier statements that would let us define v_1 and v_4 after the above equations are solved for v_2 and v_3,

$$\begin{aligned} v_1 &= \left(1 - \frac{h}{R_4}\right) v_2 && (3) \\ v_4 &= U && (4) \end{aligned}$$

2. The nodes in this case are already marked with potentials.



Extended nodal analysis

Let's define the unknown currents in the voltage sources, with the positive direction going into the source's + terminal: i_α in the independent voltage source U, and i_β in the dependent voltage source.

Write KCL (let's take outgoing currents) at all nodes except earth:

$$\text{KCL}(1): \quad 0 = \frac{v_1}{R_1} + \frac{v_1 - v_3}{R_3} - i_\alpha \tag{1}$$

$$\text{KCL}(2): \quad 0 = \frac{v_2}{R_2} + \frac{v_2 - v_4}{R_4} + i_\alpha + K_1 i_x \tag{2}$$

$$\text{KCL}(3): \quad 0 = I + \frac{v_3 - v_1}{R_3} + i_\beta \tag{3}$$

$$\text{KCL}(4): \quad 0 = \frac{v_4 - v_2}{R_4} - i_\beta. \tag{4}$$

The voltage sources introduced the problem of two extra unknowns in the above equations; they can solve this problem by providing two extra equations without further unknowns:

$$v_2 - v_1 = U \tag{5}$$

$$v_3 - v_4 = K_2\, v_2. \tag{6}$$

The controlling variables of the dependent sources need to be defined in terms of the other known or unknown quantities. Our dependent voltage source's controlling variable is the potential v_2, which is an unknown that we already introduced in the KCL equations: nothing more needs to be done for that. Our dependent current source's controlling variable is a current i_x marked in R_2. This can be described as

$$i_x = -\frac{v_2}{R_2}. \tag{7}$$

Supernode and Simplifications approach (fewer equations, more thinking)

Identify the supernodes:

Nodes 1 and 2 are joined by source U, so they can be treated as a supernode.
Let's keep potential v_1 in the circuit, and replace v_2 with $v_1 + U$.

Similiarly, nodes 3 and 4 are joined by VCVS $K_2 v_2$.
Let's keep v_4, and replace v_3 with $v_4 + K_2 v_2$.

The only remaining node is the reference node (earth), on which we do not write KCL.

$$\begin{aligned} \text{KCL}(1\&2):\ 0 &= \frac{v_1}{R_1} + \frac{v_1+U}{R_2} - \frac{K_1(v_1+U)}{R_2} + \frac{v_1-(v_4+K_2v_2)}{R_3} + \frac{(v_1+U)-v_4}{R_4} \quad (1) \\ \text{KCL}(3\&4):\ 0 &= \frac{(v_4+K_2v_2)-v_1}{R_3} + \frac{v_4-(v_1+U)}{R_4} + I \quad (2) \end{aligned}$$

In the text above, we mentioned that we'd "replace v_2 with $v_1 + U$" etc. But we should also ensure this is clearly declared in our equation system — it's essential information to let us find v_2 and v_3:

$$\begin{aligned} v_2 &= v_1 + U \quad (3) \\ v_3 &= v_4 + K_2 v_2. \quad (4) \end{aligned}$$

Together, the above set of 4 equations in the 4 unknown node potentials should be able to give our solution.

3. If you tried question 8 from this Topic's "exercises" file, then see the solution in the same file.