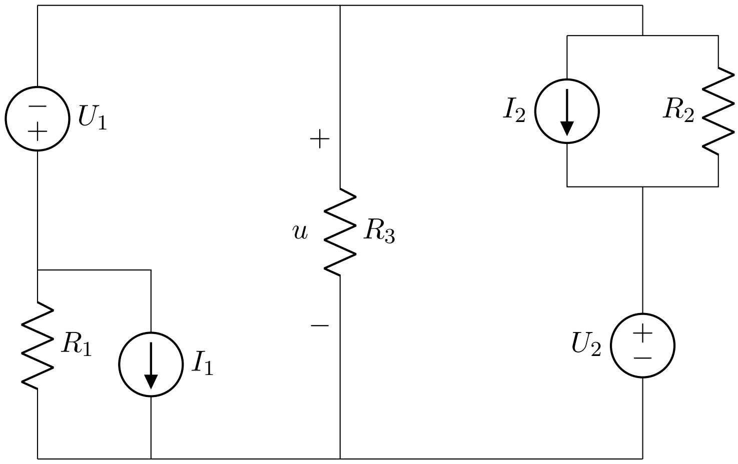
**Tutorial Questions: Superposition.**

**1.**




Find $u$.
Values: $U_1 = 6\,\text{V}$, $U_2 = 3\,\text{V}$, $I_1 = 2\,\text{A}$, $I_2 = 4\,\text{A}$, $R_1 = 24\,\Omega$, $R_2 = 24\,\Omega$, $R_3 = 12\,\Omega$.

**Superposition state (1):** $U_1$ and $U_2$ active.

One method

The zeroed current sources can be rubbed out of the diagram (open-circuits). We could write a single nodal equation (KCL) at the top or the bottom, summing the currents in the three branches, with $u_{(1)}$ as the unknown:

$$\frac{u_{(1)} + U_1}{R_1} + \frac{u_{(1)}}{R_3} + \frac{u_{(1)} - U_2}{R_2} = 0.$$

The solution of this for $u_{(1)}$ is

$$u_{(1)} = \frac{\frac{U_2}{R_2} - \frac{U_1}{R_1}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}.$$

(Notice that for this circuit we could easily have extended this analysis to solve the whole circuit without superposition, by just adding $I_1$ and $I_2$ terms to the KCL equation.)

Alternative method

We could alternatively have used source-conversion to make each series $U_n$,$R_n$ branch be a parallel current-source and resistor: then we have a circuit of just five parallel components, where current $U_2/R_2 + U_1/R_1$ passes down through parallel resistance $R_1 \parallel R_2 \parallel R_3$. This gives the same result as above.

**Superposition state (2):** $I_1$ and $I_2$ active.

With the voltage sources short-circuited, this is a similar circuit structure to the 'Alternative method' shown above: two sources $I_1 + I_2$ feed current *up* the three parallel resistors, so

$$u_{(2)} = \frac{-I_1 - I_2}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}.$$

**Total:** add the results of the superposition states.

$$u = u_{(1)} + u_{(2)} = \frac{\frac{U_2}{R_2} - \frac{U_1}{R_1} - I_1 - I_2}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}.$$

You might like to manipulate this for 'simplification', e.g.,

$$u = R_3 \frac{U_2 R_1 - U_1 R_2 - (I_1 + I_2) R_1 R_2}{R_1 R_2 + R_2 R_3 + R_3 R_1}.$$

The numeric solution with these inputs

```
U1=6; U2=3; I1=2; I2=4; R1=24; R2=24; R3=12;
```

is

```
u = R3 * ( U2*R1 - U1*R2 - (I1+I2)*R1*R2 ) / ( R1*R2 + R2*R3 + R3*R1 )
```

which comes out as `u = -36.75`, i.e. $u = -36.75\,\mathrm{V}$.

* * * *

In this circuit it's initially hard to see superposition as any advantage for getting a solution, compared to just writing the one-node nodal analysis on the complete circuit. In the above solution, superposition seems to have taken longer than direct nodal analysis.

But actually, if you like 'puzzle-thinking' you can consider how this circuit could be quite quickly solved by superposition treating every source separately: in each of the four cases, the solution is very similar (due to symmetries in the circuit) so you only have to think hard for the first case, then re-use the solution method for the others. It's particularly convenient when working with numbers, because you can just write down the numeric results from the different states, then add them, without risking creating scary long symbolic expressions to add together and simplify.

In the above circuit, the state with only $I_1$ active is similar to the state with only $I_2$ active, if we just change the subscripts 1 and 2: so solve one case, then you can write the other case by changing a few subscripts. Even when only $U_1$ or $U_2$ is active, the circuit is similar if we (in our heads) do a source-transformation and take care about the sign. Conveniently, the denominator of our solution doesn't change when we swap subscripts, as it contains every one of the resistors in an equal way. So if we saw this at an early stage, we could get the above solution quite quickly … but we won't always see the useful tricks, and they won't always work — trying nodal analysis is often a good approach if we can't see a quicker way!

Let's try it numerically, starting with a voltage-source active.

Only $U_1$ active: $R_3$ and $R_2$ form a parallel resistance of $8\,\Omega$, which is in a voltage divider with the $24\,\Omega$ resistor $R_1$. We get $u_{(\mathrm{U1})} = \frac{-1}{4} U_1 = -1.5\,\mathrm{V}$.

Only $U_2$ active: a similar situation except opposite direction and lower value of voltage source. $R_1$ and $R_2$ form a parallel resistance of $8\,\Omega$, which is in a voltage divider with the $24\,\Omega$ resistor $R_2$. We get $u_{(\mathrm{U2})} = \frac{1}{4} U_2 = 0.75\,\mathrm{V}$.

Only $I_1$ active: source-transformation of $I_1$,$R_1$ gives a circuit similar to the previous case with just $U_1$ active! We get $u_{(\mathrm{I1})} = \frac{-1}{4} I_1 R_1 = -12\,\mathrm{V}$.

Only $I_2$ active: source-transformation of $I_2$,$R_2$ gives a circuit similar to the previous case with just $U_2$ active. We get $u_{(\mathrm{I2})} = \frac{1}{4}I_2R_2 = -24\,\mathrm{V}$.

Hence, $u = (-24\,\mathrm{V}) + (-12\,\mathrm{V}) + (-1.5\,\mathrm{V}) + (0.75\,\mathrm{V})$.
$u = -36.75\,\mathrm{V}$.

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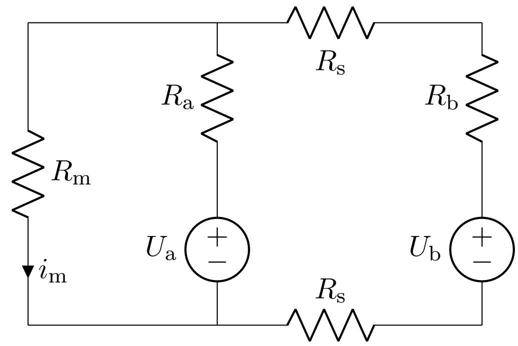
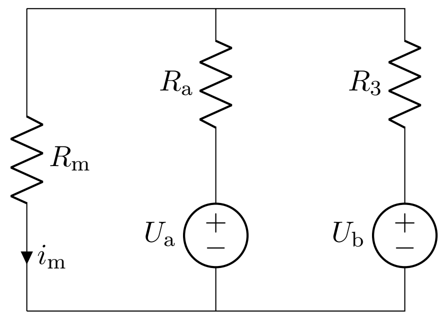
**2. Car load, car battery, help-battery!**

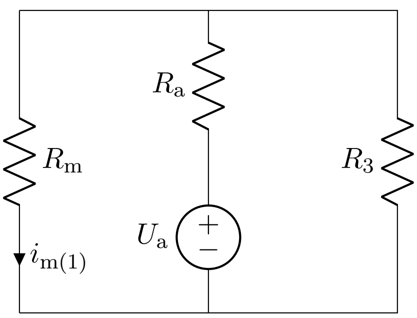
We're looking for $i_\mathrm{m}$.

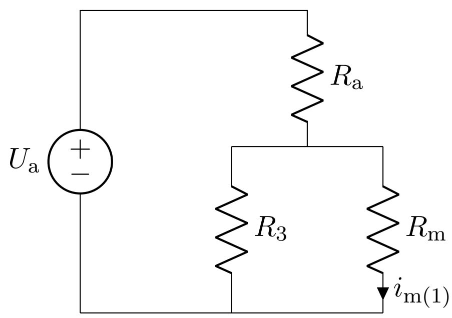



With $U_\mathrm{a}$ active and $U_\mathrm{b}$ zeroed (short-circuit) some combination of current and/or voltage division helps us. Let's call the three series resistors $R_3 = R_\mathrm{b} + 2R_\mathrm{s}$. The following circuit will still give the same solution of $i_\mathrm{m}$.




**State (1):** With $U_\mathrm{a}$ active and $U_\mathrm{b} = 0$, we get the following circuit, which is simplified on the right by redrawing.




The current through $R_\mathrm{a}$ is found by Ohm's law on the total resistance of the three resistors. Current division lets us find $i_\mathrm{m}$ from this:

$$i_{\mathrm{m}(1)} = \frac{U_\mathrm{a}}{R_\mathrm{a} + \frac{R_3R_\mathrm{m}}{R_3+R_\mathrm{m}}} \cdot \frac{R_3}{R_3 + R_\mathrm{m}} = \frac{U_\mathrm{a}R_3}{R_\mathrm{a}(R_3 + R_\mathrm{m}) + R_3R_\mathrm{m}}.$$

An alternative route is to use voltage division to find the voltage across $R_{\mathrm{m}}$, then Ohm's law to find $i_{\mathrm{m}}$:

$$i_{\mathrm{m}(1)} \;=\; \frac{U_{\mathrm{a}}\frac{R_3 R_{\mathrm{m}}}{R_3+R_{\mathrm{m}}}}{R_{\mathrm{a}}+\frac{R_3 R_{\mathrm{m}}}{R_3+R_{\mathrm{m}}}} \cdot \frac{1}{R_{\mathrm{m}}} \;=\; \frac{U_{\mathrm{a}} R_3}{R_{\mathrm{a}}(R_3+R_{\mathrm{m}})+R_3 R_{\mathrm{m}}}.$$

**State (2):** With $U_{\mathrm{b}}$ active and $U_{\mathrm{a}} = 0$, the circuit is similar in structure but has different names of the components (it has a symmetry with the previous case). So we can directly take the previous solution, swapping $U_{\mathrm{a}}$ with $U_{\mathrm{b}}$, and $R_{\mathrm{a}}$ with $R_3$.

$$i_{\mathrm{m}(2)} \;=\; \frac{U_{\mathrm{b}} R_{\mathrm{a}}}{R_3(R_{\mathrm{a}}+R_{\mathrm{m}})+R_{\mathrm{a}} R_{\mathrm{m}}}.$$

**Total:** the sum of the results from the two superposition states is helped by noticing that their denominators are the same (expand the parentheses).

$$i_{\mathrm{m}} \;=\; i_{\mathrm{m}(1)} + i_{\mathrm{m}(2)} \;=\; \frac{U_{\mathrm{a}} R_3 + U_{\mathrm{b}} R_{\mathrm{a}}}{R_{\mathrm{a}} R_{\mathrm{m}} + R_{\mathrm{a}} R_3 + R_{\mathrm{m}} R_3}$$

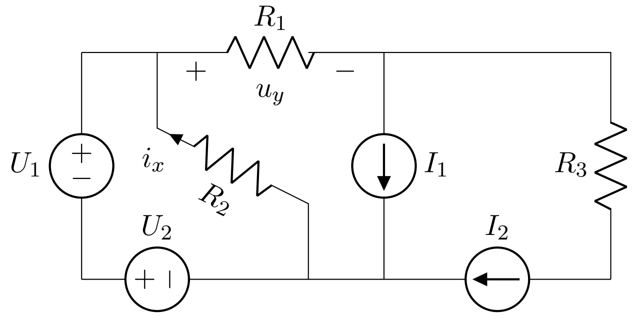
Now this must be expressed purely in terms of the given quantities: the $R_3$ that we defined must be replaced with $R_{\mathrm{b}} + 2R_{\mathrm{s}}$,

$$i_{\mathrm{m}} \;=\; \frac{U_{\mathrm{a}}(R_{\mathrm{b}}+2R_{\mathrm{s}}) + U_{\mathrm{b}} R_{\mathrm{a}}}{R_{\mathrm{a}} R_{\mathrm{m}} + (R_{\mathrm{a}}+R_{\mathrm{m}})\,(R_{\mathrm{b}}+2R_{\mathrm{s}})}.$$

We didn't expect any useful simplification by doing this, as $R_{\mathrm{b}}$ and $R_{\mathrm{s}}$ don't appear anywhere else in the expression.

**3. A practice where superposition probably doesn't make it any easier!**

Find the marked $i_x$ and $u_y$, using superposition. Two 'groups' of sources are suggested: $U_1$ and $U_2$ active, then $I_1$ and $I_2$ active. Direct solution (no superposition) by KVL, KCL and Ohm's law is almost certainly easier for this circuit.




We'll go through each of the four sources in turn, setting the others to zero. In each case we'll find the contribution of the source to $i_x$ and $u_y$. To understand this, if you're not yet very confident to do it in your head, then **draw** the diagram for each case, simplifying as much as possible.

$U_1$ active. The only remaining circuit is $U_1$ connected to $R_2$.
$i_{x(1)} = -\frac{U_1}{R_2}, \qquad u_{y(1)} = 0.$

$U_2$ active. The only remaining circuit is $U_2$ connected to $R_2$.
$i_{x(2)} = -\frac{U_2}{R_2}, \qquad u_{y(2)} = 0.$

$I_1$ active. $R_2$ is shorted by the voltage sources, so has zero voltage and thus zero current. All of current $I_1$ passes through $R_1$ (KCL).
$i_{x(3)} = 0, \qquad u_{y(3)} = I_1 R_1.$

$I_2$ active. $R_2$ is shorted by the voltage sources, so has zero voltage and thus zero current. All of current $I_2$ passes through $R_1$ (KCL).
$i_{x(4)} = 0, \qquad u_{y(4)} = I_2 R_1.$

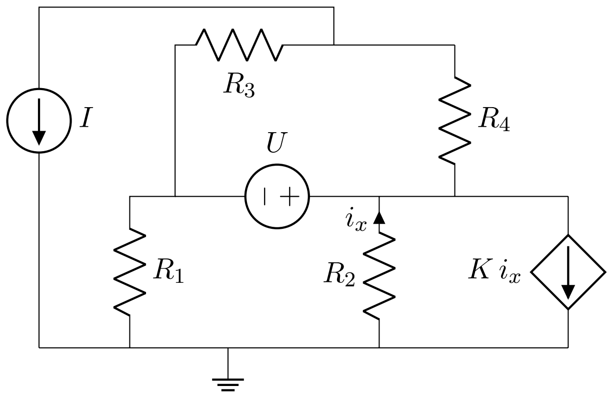
The total is then

$$i_x = \frac{-U_1}{R_2} + \frac{-U_2}{R_2} + 0 + 0 = \frac{-U_1 - U_2}{R_2}, \qquad \text{and} \qquad u_y = 0 + 0 + I_1 R_1 + I_2 R_1 = (I_1 + I_2) R_1.$$

This circuit appeared in the 'IT 2016-03' Exam, task 1: that solution can be checked against ours, above.

**4. A dependent source**

Find $i_x$ by superposition.



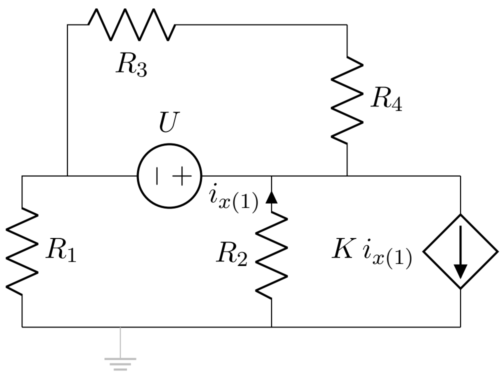

For simplicity of equations, the question defined that all of the resistances are $R$, i.e. $R_1 = R_2 = R_3 = R_4 = R$. We'll start by using the unique names such as $R_1$, to make clear which component is meant: then we'll change these to $R$ when simplifying.

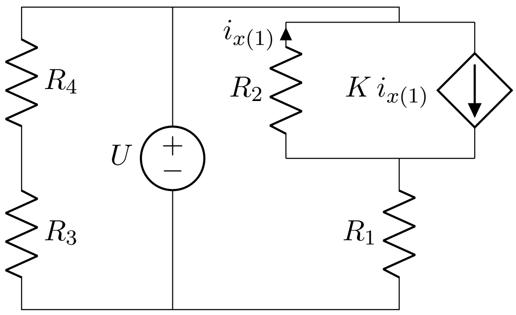
There are just two independent sources. Using superposition on this circuit therefore involves taking one of these acting alone, then the other acting alone. The dependent source is left in the circuit in both cases (the usual way to handle dependent sources in superposition).

**State 1:** Only $U$ active.

The current source is zeroed (open-circuit), so we can remove it from the diagram. We can also remove the earth symbol: it doesn't affect the sought $i_{x(1)}$ as there's only one such symbol in the circuit so no current flows in it; and it doesn't help us in doing the solution as we haven't defined any potentials.

We can then (below right) re-draw the circuit in a perhaps easier way to understand.




From this re-drawing, we see that the resistors on the left don't affect the solution of $i_{x(1)}$: the voltage source is connected in parallel with this pair of resistors, so we can reduce all of $U$, $R_3$, $R_4$ to just the source $U$ without affecting the branch where $i_{x(1)}$ is marked. To put it another way, we can write KVL around the loop $U$, $R_1$, $R_2$, without being affected by the components on the left.

By KCL, the current downward in $R_1$ is

$$i_{\mathrm{R1}} = Ki_{x(1)} - i_{x(1)} = (K-1)i_{x(1)}.$$

Taking KVL around the loop of $U$, $R_1$, $R_2$, we find

$$U - (-i_{x(1)})R_2 - (K-1)i_{x(1)}R_1 = 0,$$

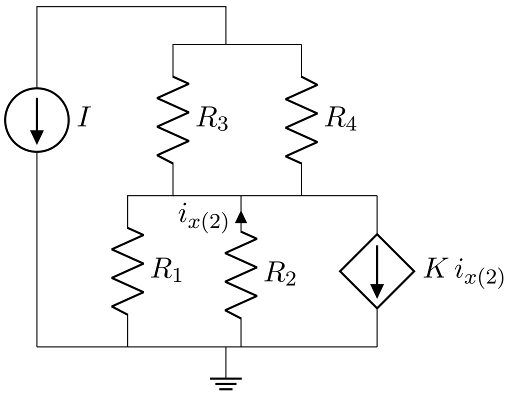
Therefore,

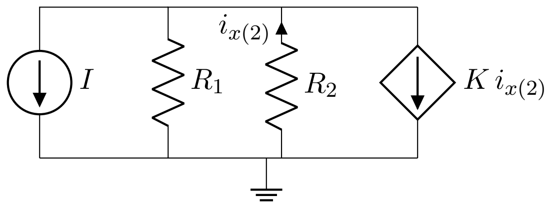
$$i_{x(1)} = \frac{U}{(K-1)R_1 - R_2} = \frac{-U}{(1-K)R_1 + R_2},$$

where the second form may be preferred for making clear that when the dependent source is weak ($K$ small) so that the resistors have the strongest influence, $i_x$ is defined against the direction that the voltage source is 'pushing'.

**State 2:** Only $I$ active.

The voltage source is zeroed (short-circuit), so we can make the nodes on both sides of it into a single node. Then the resistors $R_3$ and $R_4$ are in parallel, and this parallel combination is in series with the current source, so we can ignore their values and just write the sum of their currents as $I$.




The potential of the top node can be expressed as $-i_{x(2)}R_2$.
By KCL,

$$0 = I + \frac{-i_{x(2)}R_2}{R_1} - i_{x(2)} + Ki_{x(2)},$$

giving

$$i_{x(2)} = \frac{I}{1 + \frac{R_2}{R_1} - K} = \frac{IR_1}{(1-K)R_1 + R_2}.$$

**Total:** Summing these two superposition states to find $i_x$,

$$i_x = i_{x(1)} + i_{x(2)} = \frac{-U}{(1-K)R_1 + R_2} + \frac{IR_1}{(1-K)R_1 + R_2} = \frac{IR_1 - U}{(1-K)R_1 + R_2}.$$

In order to take advantage of the information that $R_1 = R_2 = R_3 = R_4 = R$, we substitute $R$ for the resistances,

$$i_x = \frac{I - U/R}{2 - K}.$$

You'll see this would have made several earlier steps easier too. On the other hand, our more difficult solution is also more general, as we can use it for any resistors. (I'm trying to justify the fact that, when writing the solution, I had forgotten about the '$R$'.)