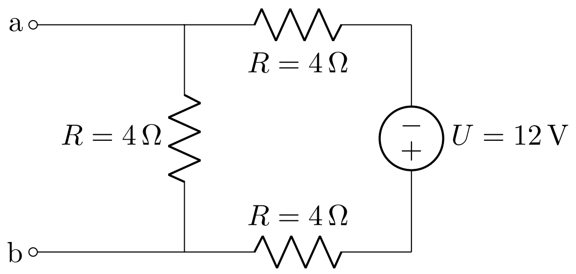
Solutions for Tutorial 04b: Thevenin/Norton Equivalent, Maximum Power

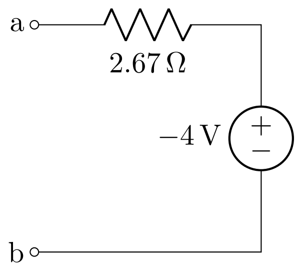
0. Quick, numeric, lab-relevant

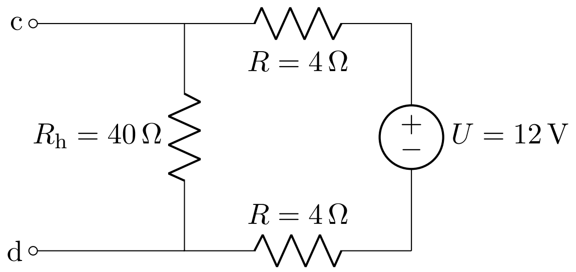
Find two-terminal equivalents (let's suggest Thevenin equivalents) of the following four circuits, between their pairs of letter-marked terminals. Be careful about directions.

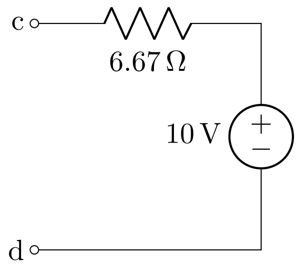
Finding open-circuit voltage then finding equivalent resistance by setting the sources to zero is probably easiest.

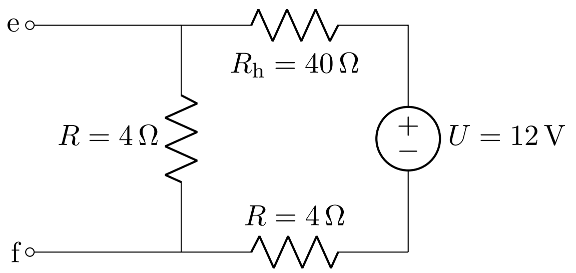
Some of the point of this is to get quicker at judging the Thevenin reistance, which can be relevant in practical situations, e.g. our old op-amp lab which used to be 'lab 2' but which has not been done since 2017. (It was relevant in lab1 also, if you think of the dividers' outputs being Thevenin sources loaded by connecting a voltmeter.)

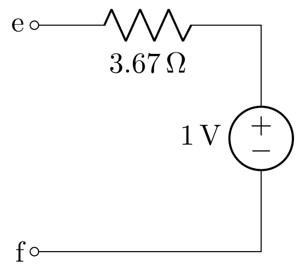


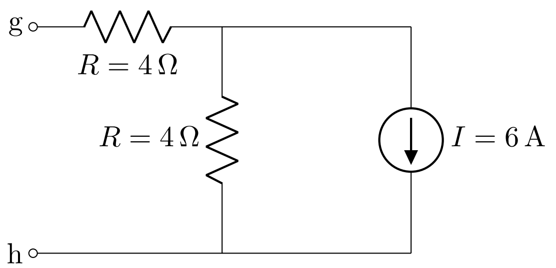


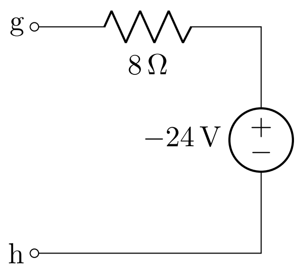










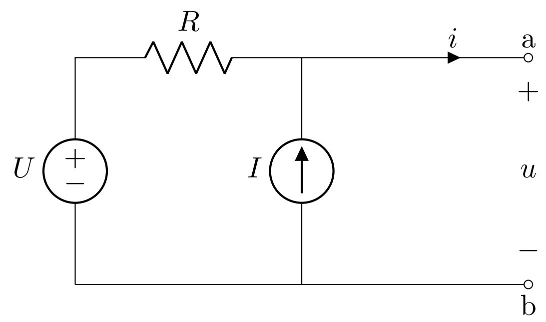


1. By finding u, i relation directly

This circuit came up in the lecture notes (2016) as an example solved by finding short-circuit and open-circuit behaviour.

Try it now by the method of finding an equation that relates u and i. This is the first method shown in the notes, but it was shown there on a more complicated circuit.

Find the Norton equivalent of this circuit between terminals a-b. Check whether you agree with the solution in the notes.



We can write KCL at the node above source I, in terms of just the unknowns u and i,

$$\frac{u-U}{R} - I + i = 0.$$

The u,i relation of a Norton source can be written as

$$i \;=\; I_{\mathrm{N}} - \frac{1}{R_{\mathrm{N}}}\,u.$$

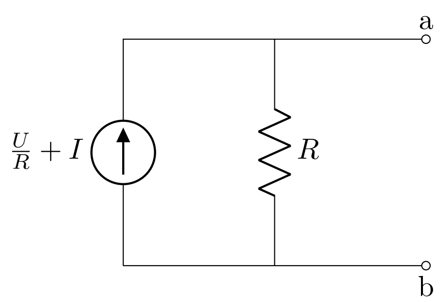
Our KCL can be rearranged to give the same form,

$$i \;=\; \left(I + \frac{U}{R}\right) - \frac{1}{R}u.$$

By comparing the coefficients in the above equation, $I_{\mathrm{N}} = I + \frac{U}{R}$, and $R_{\mathrm{N}} = R$.

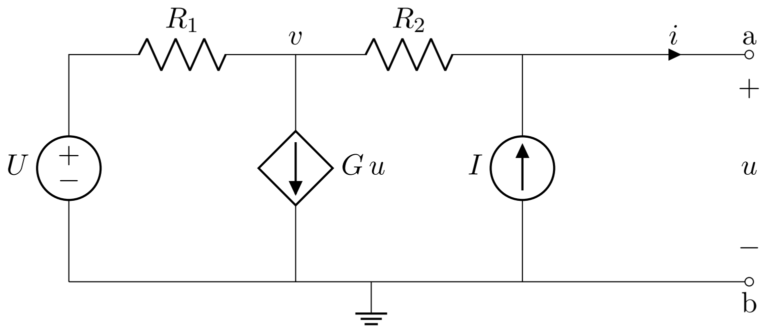
That felt actually easier than the short-circuit, open-circuit method.

To prove that we understand what a Norton source is made of, and what direction its current source must point relative to the terminals to make it equivalent to the original circuit, let's draw it.

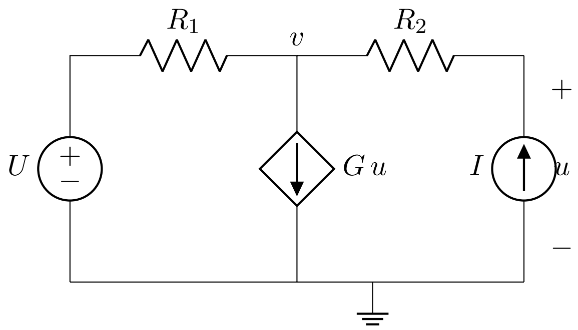


2. By short-circuit current and open-circuit voltage

This circuit also came up in the notes, as the first example. Now we'll find the Thevenin equivalent at a-b, using the method of finding i_{sc} and u_{oc}.



Open-circuit condition: find u when $i = 0$.



We have two unknown potentials, u and v.

Writing a KCL at node v,

$$\frac{v-U}{R_1} + Gu - I = 0$$

This is one equation with two unknowns. If we were looking for a u,i relation and had just u and i as unknowns, we'd be happy with a system we can't fully solve. But here we're looking at a specific case (open-circuit, $i = 0$) that has a specific solution: we need to solve for u. So a further equation is needed to let us eliminate v.

Nodal analysis tells us to write another KCL, at the other node with unknown potential,

$$\frac{u-v}{R_2} - I = 0.$$

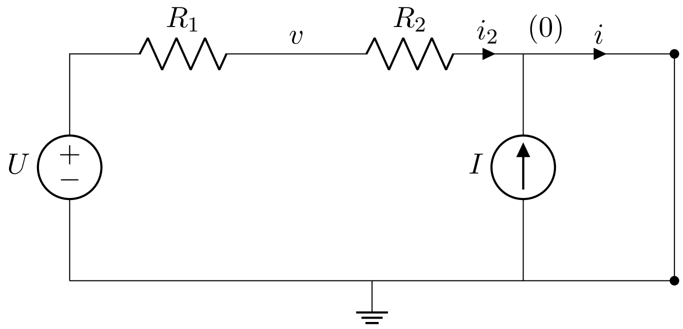
Using this, we can substitute $v = u - IR_2$ in the earlier equation to eliminate v.

$$\frac{(u-IR_2)-U}{R_1} + Gu - I \;=\; 0 \qquad \Longrightarrow \qquad u\left(\frac{1}{R_1}+G\right) = \frac{U}{R_1} + I\left(1+\frac{R_2}{R_1}\right)$$

giving the open-circuit voltage as

$$u \;=\; \frac{\frac{U}{R_1} + I\left(1+\frac{R_2}{R_1}\right)}{\frac{1}{R_1}+G} \;=\; \frac{U + I(R_1+R_2)}{1+GR_1}.$$

Short-circuit conditions: find i when $u = 0$.



We have fixed $u = 0$, so the dependent source Gu has zero output. Being a zeroed current source, it can be removed (open-circuit) from the diagram.

By KVL around the outer loop, and Ohm's law, the current from left to right in R_2 is

$$i_2 = \frac{U}{R_1 + R_2}.$$

By KCL above the current source,

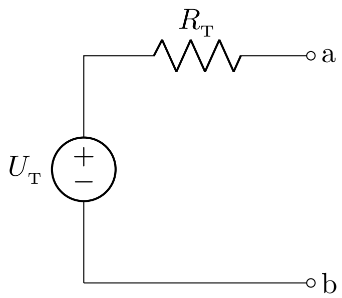
$$i = I + i_2 = I + \frac{U}{R_1 + R_2}.$$

Total

Now that we have the short-circuit current and open-circuit voltage, the Thevenin equivalent can be written.

$$U_{\text{T}} = u_{\text{oc}} = \frac{U + I(R_1 + R_2)}{1 + GR_1}.$$

$$R_{\text{T}} = \frac{u_{\text{oc}}}{i_{\text{sc}}} = \frac{\frac{U+I(R_1+R_2)}{1+GR_1}}{I + \frac{U}{R_1+R_2}} = \frac{R_1 + R_2}{1 + GR_1}.$$



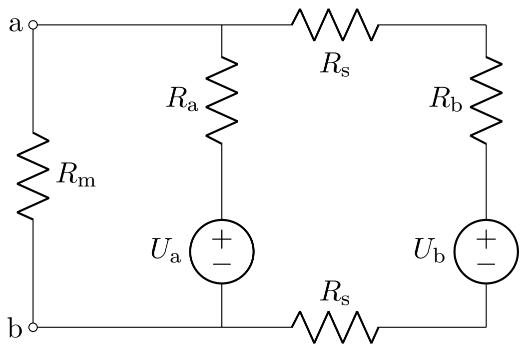
3. Maximum power

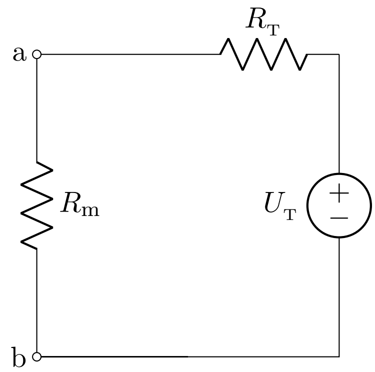
What value of R_m (as a function of U and R) will result in the highest possible power being delivered to R_m?

What is the delivered power to R_m in this case?

To simplify the final expression, we're given that $U_\mathrm{a} = U_\mathrm{b} = U$ and $R_\mathrm{a} = R_\mathrm{b} = R_\mathrm{s} = R$.

This is definitely a "maximum power" question! If we find an equivalent (I seem to choose Thevenin equivalents by preference) for rest of the circuit (outside R_m) then we can directly apply the results from maximum power theory to find that $R_\mathrm{m} = R_{\mathrm{T}}$ maximises the power transfer to R_{m}, and to find this power as $U_{\mathrm{T}}^2/4R_{\mathrm{m}}$.





The open-circuit voltage at a-b (R_m not connected) can be found by KVL in the loop,

$$u_{\mathrm{ab(oc)}} = U_{\mathrm{T}} = U_\mathrm{a} + \frac{U_\mathrm{b} - U_\mathrm{a}}{R_\mathrm{a} + R_\mathrm{b} + 2R_\mathrm{s}}\, R_\mathrm{a} = \frac{U_\mathrm{a}(R_\mathrm{b} + 2R_\mathrm{s}) + U_\mathrm{b}R_\mathrm{a}}{R_\mathrm{a} + R_\mathrm{b} + 2R_\mathrm{s}}$$

The source resistance can be found by setting the sources to zero and simplifying the resistors to a single resistance; these resistors are R_a in parallel with the series branch of R_b and two R_s.

$$R_{\mathrm{T}} = \frac{R_\mathrm{a}\left(R_\mathrm{b} + 2R_\mathrm{s}\right)}{R_\mathrm{a} + R_\mathrm{b} + 2R_\mathrm{s}}$$

By the maximum power theorem, the power into R_m (if we can choose R_m but cannot change the rest of the circuit) is obtained if $R_\mathrm{m} = R_\mathrm{T}$.

The actual power in that case is

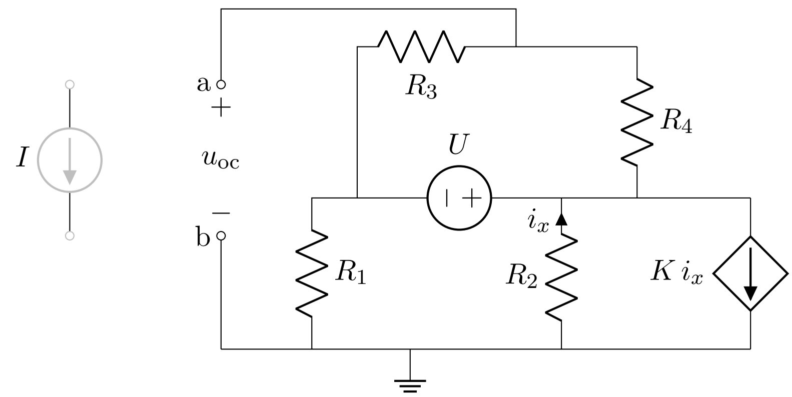
$$P_{(\mathrm{max})} = \frac{U_{\mathrm{T}}^2}{4R_{\mathrm{T}}} = \frac{\left(U_\mathrm{a}(R_\mathrm{b} + 2R_\mathrm{s}) + U_\mathrm{b}R_\mathrm{a}\right)^2}{4R_\mathrm{a}\left(R_\mathrm{b} + 2R_\mathrm{s}\right)\left(R_\mathrm{a} + R_\mathrm{b} + 2R_\mathrm{s}\right)}$$

On seconds thoughts, finding short-circuit current and a Norton-source would probably have simplified the arithmetic a bit in this question.

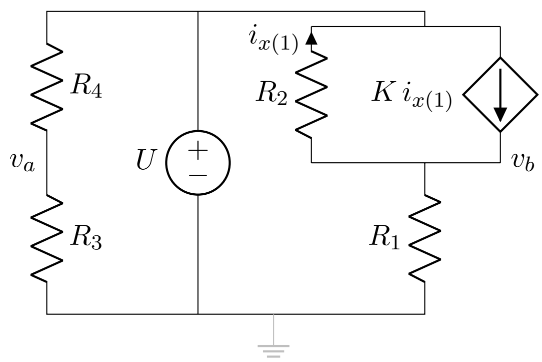
4. A dependent source

Here's another familiar circuit, that came up in the superposition exercises. Again, we define R, and let $R_1 = R_2 = R_3 = R_4 = R$.

"Find what value of source I should be chosen in order to maximize the power supplied *to* this current source from the rest of the circuit."



By a similar re-drawing to the one used for superposition (previous set of exercises) we can get the following circuit, where $v_a - v_b$ is the same as u_{oc} (marked above). Notice that the reference node (earth) has been moved in order to help our thinking: this choice of a reference does not affect the u,i behaviour at the terminals a-b.[1]



From this re-drawing, we see that the resistors on the left don't affect the solution of $i_{x(1)}$: the voltage source is connected in parallel with this pair of resistors, so we can reduce all of U, R_3, R_4 to just the source U without affecting the branch where $i_{x(1)}$ is marked. To put it another way, we can write KVL around the loop U, R_1, R_2, without being affected by the components on the left.

By KCL, the current downward in R_1 is

$$i_{\mathrm{R1}} = Ki_{x(1)} - i_{x(1)} = (K-1)i_{x(1)}.$$

[1] Later, with opamps, we see that several earth symbols can be used in a circuit, and are in that case assumed to be connected together, i.e. all the same node, so currents can flow between them. A 'two-terminal equivalent' is based on there only being two nodes connecting between the part that's an equivalent, and the rest of a circuit. If there are earth nodes that let current move between the inside and outside of the equivalenced part, then these are already one of the nodes … we can then define only one (not two) more connections if we want to use 'two-terminal equivalent' methods.

Taking KVL around the loop of U, R_1, R_2, we find

$$U - (-i_{x(1)})R_2 - (K-1)i_{x(1)}R_1 = 0,$$

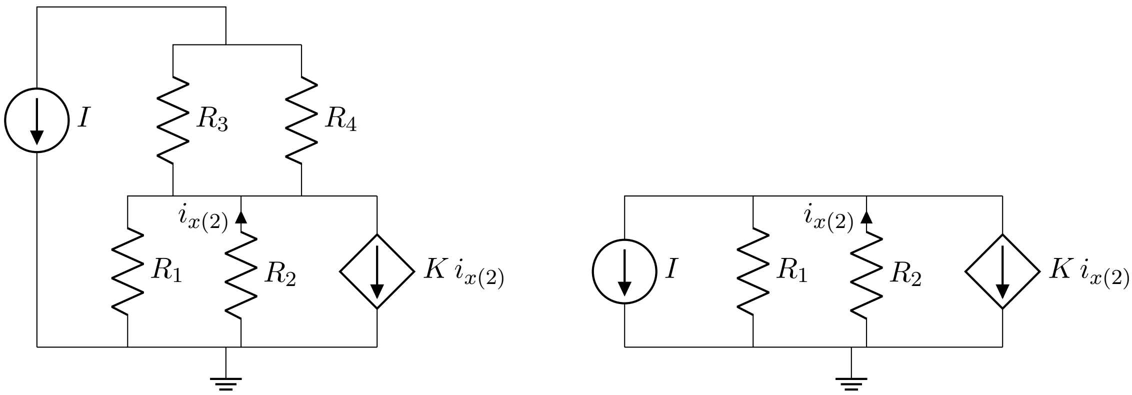
Therefore,

$$i_{x(1)} \;=\; \frac{U}{(K-1)R_1 - R_2} \;=\; \frac{-U}{(1-K)R_1 + R_2},$$

where the second form may be preferred for making clear that when the dependent source is weak (K small) so that the resistors have the strongest influence, i_x is defined against the direction that the voltage source is 'pushing'.

State 2: Only I active.

The voltage source is zeroed (short-circuit), so we can make the nodes on both sides of it into a single node. Then the resistors R_3 and R_4 are in parallel, and this parallel combination is in series with the current source, so we can ignore their values and just write the sum of their currents as I.



The potential of the top node can be expressed as $-i_{x(2)}R_2$.
By KCL,

$$0 = I + \frac{-i_{x(2)}R_2}{R_1} - i_{x(2)} + Ki_{x(2)},$$

giving

$$i_{x(2)} \;=\; \frac{I}{1 + \frac{R_2}{R_1} - K} \;=\; \frac{IR_1}{(1-K)R_1 + R_2}.$$

Total: Summing these two superposition states to find i_x,

$$i_x = i_{x(1)} + i_{x(2)} \;=\; \frac{-U}{(1-K)R_1 + R_2} + \frac{IR_1}{(1-K)R_1 + R_2} \;=\; \frac{IR_1 - U}{(1-K)R_1 + R_2}.$$